Tensor Representation of Spin States

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(Received 12 September 2014; published 25 February 2015)

We propose a generalization of the Bloch sphere representation for arbitrary spin states. It provides a compact and elegant representation of spin density matrices in terms of tensors that share the most important properties of Bloch vectors. Our representation, based on covariant matrices introduced by Weinberg in the context of quantum field theory, allows for a simple parametrization of coherent spin states, and a straightforward transformation of density matrices under local unitary and partial tracing operations. It enables us to provide a criterion for anticoherence, relevant in a broader context such as quantum polarization of light.

DOI: 10.1103/PhysRevLett.114.080401 PACS numbers: 03.65.Aa, 03.65.Ca

The concept of spin is ubiquitous in quantum theory and all related fields of research, such as solid-state physics, molecular, atomic, nuclear or high-energy physics [1–5]. It has profound implications for the structure of matter as a consequence of the celebrated spin-statistics theorem [6]. The spin of a quantum system, be it an electron, a nucleus, or an atom, has also been proven to be a key resource for many applications such as in spintronics [7], quantum information theory [8], or nuclear magnetic resonance [9]. Simple geometrical representations of spin states [10] allow one to develop physical insight regarding their general properties and evolution. Particularly well studied is the case of a single two–level system, formally equivalent to a spin-$1/2$. In this case, the geometric representation is particularly simple. Indeed, the density matrix can be expressed in a basis formed of Pauli matrices and the identity matrix, leading to a parametrization in terms of a vector in $\mathbb{R}^3$. Pure states correspond to points on a unit sphere, the so-called Bloch sphere, and mixed states fill the inside of the sphere, the "Bloch ball". The simplicity of this representation help visualize the action and geometry of all possible spin-$1/2$ quantum channels [11]. For arbitrary pure spin states, another nice geometrical representation has been developed by Majorana in which a spin-j state is visualized as $2j$ points on the Bloch sphere [12]. This so-called Majorana or stellar representation has been exploited in various contexts (see, e.g., [11,13–17]), but cannot be generalized to mixed spin states.

Given the importance of geometrical representations, there have been numerous attempts to extend the previous representations to arbitrary mixed states. The former rely on a variety of sophisticated mathematical concepts such as $su(N)$-algebra generators [10,18,19], polarization operator basis [20–22], Weyl operator basis [23], quaternions [24], octonions [25], or Clifford algebra [26]. In the present Letter, we propose an elegant generalization to arbitrary spin-j of the spin-$1/2$ Bloch sphere representation based on matrices introduced by Weinberg in the context of quantum field theory [27]. The main result of the Letter is Theorem 2, which allows us to express any spin-j density matrix as a linear combination of matrices with convenient properties. The remarkable features of our representation are especially reflected in the simple coordinates of coherent states, transformation under SU(2) operations, and the simplicity of the representation of reduced density matrices. To illustrate the usefulness of such a representation, we show that it allows us to give an easy characterization of anticoherent spin states. Such states have been studied in various contexts, such as quantum polarization of light (see, e.g., [28,29]), spherical designs [30], as well as in the search for maximally entangled symmetric states [31]. We believe that our representation should prove useful in many of the contexts where the spin formalism is used.

We construct this parametrization of a spin-j density matrix ρ from the set of 4^{2j} covariant matrices [27]. Defining the four-vector $q = (q_0, q_1, q_2, q_3) \equiv (q_0, \mathbf{q})$, Weinberg's covariant matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$, with $0 \le \mu_i \le 3$, are constructed from products of components of $\mathbf{J} = (J_1, J_2, J_3)$ with J_a $(1 \le a \le 3)$, the usual $(2j+1)$-dimensional representations of angular momentum operators. They can be obtained by expanding the square of the $(2j+1)$-dimensional matrix corresponding to the $(j,0)$ representation of a Lorentz boost in direction $\mathbf{q}$, which can be put in the form [27]

$$\Pi^{(j)}(q) \equiv (q_0^2 - |\mathbf{q}|^2)^j e^{-2\eta_q \hat{\mathbf{q}}\cdot\mathbf{J}}, \tag{1}$$

with $\eta_q = \operatorname{arctanh}(-|\mathbf{q}|/q_0)$ and $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$. Matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$ are defined in [27] by identifying the coefficients of the multivariate polynomial with variables q_0, q_1, q_2, q_3 in Eq. (1) with those of the polynomial

$$\Pi^{(j)}(q) = (-1)^{2j} q_{\mu_1} q_{\mu_2} \ldots q_{\mu_{2j}} S_{\mu_1\mu_2\ldots\mu_{2j}} \tag{2}$$

(we use Einstein summation convention for repeated indices). An explicit expression for $\Pi^{(j)}(q)$ is given in [27] as

$$\Pi^{(j)}(q) = (q_0^2 - \mathbf{q}^2)^j + \sum_{k=1}^{j} \frac{(q_0^2 - \mathbf{q}^2)^{j-k}}{(2k)!} (2\mathbf{q}\cdot\mathbf{J}) \left(\prod_{r=1}^{k-1} [(2\mathbf{q}\cdot\mathbf{J})^2 - (2r\mathbf{q})^2] \right) (2\mathbf{q}\cdot\mathbf{J} + 2kq_0) \quad (3)$$

for integer j, and

$$\Pi^{(j)}(q) = (q_0^2 - \mathbf{q}^2)^{j-1/2}(-q_0 - 2\mathbf{q}\cdot\mathbf{J}) - \sum_{k=1}^{j-1/2} \frac{(q_0^2 - \mathbf{q}^2)^{j-1/2-k}}{(2k+1)!} \left(\prod_{r=1}^{k} [(2\mathbf{q}\cdot\mathbf{J})^2 - (2r\mathbf{q} - \mathbf{q})^2] \right) [2\mathbf{q}\cdot\mathbf{J} + (2kq_0 + q_0)] \quad (4)$$

for half-integer j. The identity matrix is implicit in front of constant terms. For instance, identifying the coefficient of q_0^{2j} in these expressions, we get that $S_{00\ldots0}$ is the $(2j+1)$-dimensional identity matrix, $\mathbb{1}_{2j+1}$. The matrices $S_{\mu_1\mu_2\ldots\mu_{2j}}$ are Hermitian matrices, invariant under permutation of indices, and they obey the following linear relation:

$$g_{\mu_1\mu_2} S_{\mu_1\mu_2\ldots\mu_{2j}} = 0, \quad (5)$$

where $g \equiv \mathrm{diag}(-,+,+,+)$.

Let us briefly consider the simplest examples. From Eq. (4), the explicit expression of $\Pi^{(j)}(q)$ for spin-$1/2$ reads

$$\Pi^{(1/2)}(q) = -q_0 - 2\mathbf{q}\cdot\mathbf{J}, \quad (6)$$

where J_a are spin-$1/2$ representations of the angular momentum operators. Identifying with Eq. (2) directly gives $S_0 = \sigma_0$ and $S_a = 2J_a = \sigma_a$, where σ_0 is the 2×2 identity matrix and σ_a are the usual Pauli matrices. The usual Bloch sphere representation for an arbitrary spin-$1/2$ density matrix $\rho = \frac{1}{2}\sigma_0 + \frac{1}{2}\mathbf{x}\cdot\boldsymbol{\sigma}$ can then be expressed in terms of the S_{μ_1} $(0 \le \mu_1 \le 3)$ as

$$\rho = \frac{1}{2} x_{\mu_1} S_{\mu_1} \quad (7)$$

with the Bloch vector $\mathbf{x} = \mathrm{tr}(\rho\boldsymbol{\sigma})$ and $x_0 = 1$.

For $j = 1$, the equality between Eqs. (2) and (3) for $\Pi^{(1)}(q)$ reads

$$(q_0^2 - \mathbf{q}^2) + 2\mathbf{q}\cdot\mathbf{J}(\mathbf{q}\cdot\mathbf{J} + q_0) = q_{\mu_1} q_{\mu_2} S_{\mu_1\mu_2}. \quad (8)$$

Identifying coefficients of this quadratic form yields $S_{00} = J_0$, $S_{a0} = J_a$ and $S_{ab} = J_aJ_b + J_bJ_a - \delta_{ab}J_0$ with J_0 the 3×3 identity matrix. Again, the set of $S_{\mu_1\mu_2}$ matrices can serve to express any spin-1 density matrix ρ as

$$\rho = \frac{1}{4} x_{\mu_1\mu_2} S_{\mu_1\mu_2} \quad (9)$$

with coordinates

$$x_{\mu_1\mu_2} = \mathrm{tr}(\rho S_{\mu_1\mu_2}). \quad (10)$$

Equations (3) and (4) can be used to generalize this expansion to arbitrary j, as we will show in Theorem 2. The main property of the covariant matrices is given by Theorem 1 below. We first give a useful lemma.

Lemma 1: Let $|\alpha\rangle$ be a spin-j coherent state, defined for $\alpha = e^{-i\varphi}\cot(\theta/2)$ with $\theta \in [0,\pi]$ and $\varphi \in [0, 2\pi[$ by

$$|\alpha\rangle = \sum_{m=-j}^{j} \sqrt{\binom{2j}{j+m}} \left[\sin\frac{\theta}{2}\right]^{j-m} \left[\cos\frac{\theta}{2} e^{-i\varphi}\right]^{j+m} |j,m\rangle \quad (11)$$

in the standard angular momentum basis $\{|j,m\rangle : -j \le m \le j\}$, and let $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Then

$$\langle\alpha|\Pi^{(j)}(q)|\alpha\rangle = (-1)^{2j}(q_0 + \mathbf{q}\cdot\mathbf{n})^{2j}. \quad (12)$$

The proof of this lemma is based on the SU(2) disentangling theorem and can be found in the Supplemental Material [32]. One of its consequences is that, by identifying coefficients of the polynomial in q_μ in Eq. (12), we get

$$\langle\alpha|S_{\mu_1\mu_2\ldots\mu_{2j}}|\alpha\rangle = n_{\mu_1} n_{\mu_2} \ldots n_{\mu_{2j}}, \quad (13)$$

with $n_0 = 1$.

In the Majorana representation, any pure spin-j state is viewed as a permutation symmetric state of a system of $N \equiv 2j$ spin-$1/2$, or equivalently, as an N-qubit symmetric state. The Hilbert space $\mathcal{H} \equiv \mathbb{C}^{2^N}$ of an N spin-$1/2$ system has dimension 2^N but its symmetric subspace $\mathcal{H}_S$ has only dimension $N + 1 = 2j + 1$. It is spanned by the symmetric Dicke states

$$|D_N^{(k)}\rangle = \mathcal{N} \sum_{\pi} |\underbrace{\downarrow\ldots\downarrow}_{N-k}\underbrace{\uparrow\ldots\uparrow}_{k}\rangle, \qquad k = 0,\ldots,N, \quad (14)$$

where the sum runs over all permutations of the string with $N-k$ spin down and k spin up, and $\mathcal{N}$ is the normalization constant. The Dicke state $|D_N^{(k)}\rangle$ corresponds to $|j,m\rangle$ with $j = N/2$ and $m = k - N/2$.

Let $\mathcal{L}(\mathcal{H})$ be the Hilbert space of linear operators acting on the finite-dimensional space $\mathcal{H}$. An operator basis for $\mathcal{L}(\mathcal{H})$ equipped with the standard Hilbert-Schmidt inner product is given by the set of the 4^N generalized Pauli matrices defined as the N-fold tensor products of the 2×2 matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ [8],

$$\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N} = \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \ldots \otimes \sigma_{\mu_N}. \quad (15)$$

These Hermitian operators verify the relations $\mathrm{tr}(\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}\boldsymbol{\sigma}_{\nu_1\nu_2\ldots\nu_N}) = 2^N\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}\ldots\delta_{\mu_N\nu_N}$ and thus form an orthogonal basis. Any state ρ of N spin-1/2 can be expanded in this basis as

$$\rho = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N}\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}, \tag{16}$$

where $x_{\mu_1\mu_2\ldots\mu_N}$ are real coefficients given by

$$x_{\mu_1\mu_2\ldots\mu_N} = \mathrm{tr}(\rho\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}). \tag{17}$$

We can now prove the following theorem:

Theorem 1: The Weinberg covariant matrices defined in Eq. (2) are given by the projection of tensor products of Pauli matrices into the subspace $\mathcal{H}_S$ of states that are invariant under permutation of particles. Namely, denoting by $\mathcal{P}_S \equiv \sum_{k=0}^{N} |D_N^{(k)}\rangle\langle D_N^{(k)}|$ the projector onto $\mathcal{H}_S$, the $S_{\mu_1\mu_2\ldots\mu_N}$ matrix corresponds to the $(N+1)$-dimensional block spanned by the $|D_N^{(k)}\rangle$ of the matrix $\mathcal{P}_S\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}\mathcal{P}_S^\dagger$, i.e., in terms of matrix elements

$$\langle D_N^{(k)}|S_{\mu_1\mu_2\ldots\mu_N}|D_N^{(\ell)}\rangle = \langle D_N^{(k)}|\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}|D_N^{(\ell)}\rangle, \tag{18}$$

with $0 \le k, \ell \le N$.

Proof.—Let $\tilde{S}_{\mu_1\mu_2\ldots\mu_N} = \mathcal{P}_S\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}\mathcal{P}_S^\dagger$. Any spin-$j$ coherent state $|\alpha\rangle$ defined by Eq. (11) can also be written as the tensor product of identical spin-1/2 coherent states. As a symmetric state, $|\alpha\rangle$ is invariant under $\mathcal{P}_S$, i.e., $|\alpha\rangle = \mathcal{P}_S|\alpha\rangle$, so that $\langle\alpha|\tilde{S}_{\mu_1\mu_2\ldots\mu_N}|\alpha\rangle = \langle\alpha|\boldsymbol{\sigma}_{\mu_1\mu_2\ldots\mu_N}|\alpha\rangle = n_{\mu_1}n_{\mu_2}\ldots n_{\mu_N}$. Using Eq. (13), we thus have

$$\langle\alpha|\tilde{S}_{\mu_1\mu_2\ldots\mu_N}|\alpha\rangle = \langle\alpha|S_{\mu_1\mu_2\ldots\mu_N}|\alpha\rangle \tag{19}$$

for all α; i.e., the Husimi functions of the two operators are identical. Therefore, $S_{\mu_1\mu_2\ldots\mu_N}$ and $\tilde{S}_{\mu_1\mu_2\ldots\mu_N}$ coincide in $\mathcal{H}_S$. □

In other words, instead of obtaining the Weinberg matrices from the expansion of the rather complicated Eqs. (3) and (4), we can construct them simply by projecting the corresponding tensor product of Pauli operators into the symmetric subspace. In order to fully exploit the consequences of this fact, we need some basic notions of frame theory [35].

A family of vectors $|\phi_i\rangle$, $i \in \{1, \ldots, M\}$, is called a frame for a Hilbert space $\mathcal{H}$ with bounds $A, B \in]0, \infty[$, if

$$A\|\psi\|^2 \le \sum_{i=1}^{M} |\langle\psi|\phi_i\rangle|^2 \le B\|\psi\|^2,\ \forall|\psi\rangle \in \mathcal{H}. \tag{20}$$

If $A = B$, then the frame is called an A-tight frame.

Orthonormal bases are a special case of A-tight frames. In particular, the generalized Pauli matrices [Eq. (15)] form—up to normalization—an orthonormal basis of $\mathcal{L}(\mathcal{H})$, and are in fact an A-tight frame, which verifies Eq. (20) with $A = B = 2^N$ and $M = 4^N$. According to Proposition 22 in [35], a frame of a Hilbert space $\mathcal{H}$ with bounds A, B that is orthogonally projected to a subspace $P\mathcal{H}$ is a frame of $P\mathcal{H}$ with the same bounds A, B. Therefore, we have as a corollary of Theorem 1 that the set of covariant matrices $S_{\mu_1\mu_2\ldots\mu_N}$ forms a 2^N-tight frame for $\mathcal{L}(\mathcal{H}_S)$.

Tight frames are in a sense a generalization of orthonormal bases, as they allow an expansion over the elements of the frames with the same formulas as for an orthonormal basis; i.e., for all $|\psi\rangle \in \mathcal{H}$, we have $|\psi\rangle = A^{-1}\sum_{i=1}^{M}\langle\phi_i|\psi\rangle|\phi_i\rangle$ (proposition 20 in [35]). This immediately entails the following result, which provides a generalization of the Bloch sphere representation for spin-1/2, Eq. (7), to any spin:

Theorem 2: For general spin-j, the 4^N Hermitian matrices $S_{\mu_1\mu_2\ldots\mu_N}$ (with $N \equiv 2j$) provide an overcomplete basis (more precisely, a 2^N-tight frame) over which ρ can be expanded, that is, any state can be expressed as

$$\rho = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N} S_{\mu_1\mu_2\ldots\mu_N}, \tag{21}$$

with coefficients

$$x_{\mu_1\mu_2\ldots\mu_N} = \mathrm{tr}(\rho S_{\mu_1\mu_2\ldots\mu_N}), \tag{22}$$

real and invariant under permutation of the indices.

Since $S_{00\ldots0}$ is the identity matrix, the condition $\mathrm{tr}\rho = 1$ for density matrices is equivalent to $x_{00\ldots0} = 1$. The tight frame property allows one to write the Hilbert-Schmidt scalar product of any two Hermitian operators ρ and ρ' with coordinates $x_{\mu_1\mu_2\ldots\mu_N}$ and $x'_{\mu_1\mu_2\ldots\mu_N}$ as the scalar product of coordinates, more precisely

$$\mathrm{tr}(\rho\rho') = \frac{1}{2^N} x_{\mu_1\mu_2\ldots\mu_N} x'_{\mu_1\mu_2\ldots\mu_N}. \tag{23}$$

The condition $\mathrm{tr}\rho^2 \le 1$ that every state must satisfy translates into $\sum_{\mu_1}\ldots\sum_{\mu_N} x^2_{\mu_1\mu_2\ldots\mu_N} \le 2^N$. Note that from Eq. (22) and the definition of $S_{\mu_1\mu_2\ldots\mu_N}$, the coordinates $x_{\mu_1\mu_2\ldots\mu_N}$ appear as the coefficients of $(-1)^N\langle\Pi^{(j)}(q)\rangle$, which is a multivariate polynomial in variables q_0, q_1, q_2, q_3,

$$(-1)^N\langle\Pi^{(j)}(q)\rangle = x_{\mu_1\mu_2\ldots\mu_N} q_{\mu_1}\ldots q_{\mu_N}. \tag{24}$$

Due to the overcompleteness of the $S_{\mu_1\mu_2\ldots\mu_N}$ the coordinates $x_{\mu_1\mu_2\ldots\mu_N}$ in Eq. (21) are so far not unique. However, for a given spin-j density matrix ρ, Eq. (22) is the unique choice of coordinates $x_{\mu_1\mu_2\ldots\mu_N}$ such that these coordinates are real numbers, invariant under permutation of the indices, and verifying the condition $g_{\mu_1\mu_2}x_{\mu_1\mu_2\ldots\mu_N} = 0$ (see Proposition 1 in the Supplemental Material [32]).

The generalized Bloch representation Eq. (21) shares with the Bloch representation of a spin-1/2 several crucial properties. First of all, using Eqs. (13) and (22), we see that coordinates of a coherent state are simply given by the product of components of the 4-vector $n = (1, \mathbf{n})$, namely $x_{\mu_1\mu_2\ldots\mu_N} = n_{\mu_1} n_{\mu_2} \ldots n_{\mu_N}$. This generalizes the fact that the Bloch vector representing a spin-1/2 state points in the direction given by the angles defining the coherent state. Secondly, under any SU(2) transformation, the Bloch vector of a spin-1/2 simply rotates, i.e., transforms according to $x_a \to R_{ab} x_b$, where R is a rotation matrix. Similarly, for higher spins the tensor of coordinates of an arbitrary state transforms according to $x_{\mu_1\ldots\mu_N} \to R_{\mu_1\nu_1} \ldots R_{\mu_N\nu_N} x_{\nu_1\ldots\nu_N}$, with R_{ab} the 3×3 rotation matrix and $R_{0\mu} = R_{\mu 0} = \delta_{\mu 0}$. This is a consequence of a more general covariance property of the basis matrices $S_{\mu_1\mu_2\ldots\mu_N}$. Indeed, they were constructed in such a way that for any element Λ of the Lorentz group, with $D^{(j)}[\Lambda]$ the $(2j+1)$-dimensional matrix associated with Λ in the $(j, 0)$ representation,

$$D^{(j)}[\Lambda] S_{\mu_1\mu_2\ldots\mu_N} D^{(j)}[\Lambda]^\dagger = \Lambda_{\mu_1}^{\nu_1} \ldots \Lambda_{\mu_N}^{\nu_N} S_{\nu_1\nu_2\ldots\nu_N} \tag{25}$$

in the covariant-contravariant notation of [27]. From Eq. (22), this property translates to coordinates $x_{\mu_1\mu_2\ldots\mu_N}$. For rotations $R_{\mu\nu}$, the distinction between upper and lower indices becomes irrelevant.

In addition to the shared advantages of a Bloch vector, our generalized Bloch sphere representation [Eq. (21)] enjoys additional convenient properties relevant for systems made of many spin-1/2 or qubits. For instance, coordinates of the spin-k reduced density matrix obtained by tracing the spin-j matrix over $j-k$ spins are simply given by

$$x_{\mu_1\ldots\mu_{2k}} = x_{\mu_1\ldots\mu_{2k}0\ldots 0} \tag{26}$$

(see Proposition 3 in the Supplemental Material [32]). Note that in [36], a similar property was observed for the coefficients in the expansion of ρ over generalized Pauli matrices, and a formal Lorentz invariance of that expansion was used very recently to generalize monogamy relations of entanglement [37].

We now consider a few examples of states and give their coordinates in our representation. The maximally mixed state $\rho_0 = 1/(2j+1)\mathbb{1}_{2j+1}$ has coordinates $x_{\mu_1\mu_2\ldots\mu_{2j}}$ given by

$$x_{\mu_1\mu_2\ldots\mu_{2j}} q_{\mu_1} \ldots q_{\mu_{2j}} = \sum_{k=0}^{j} \frac{\binom{2j}{2k}}{2k+1} q_0^{2(j-k)} |\mathbf{q}|^{2k} \tag{27}$$

(see Proposition 2 in the Supplemental Material [32]). Another example is given by the Schrödinger cat states $|\psi_{\mathrm{cat}}^{(j)}\rangle = (|j, -j\rangle + |j, j\rangle)/\sqrt{2}$. By linearity of Eq. (21) and of the trace, they have coordinates

$$x_{\mu_1\ldots\mu_N}^{\mathrm{cat}} = \frac{1}{2}\left[\prod_{i=1}^{N} n_{\mu_i}^{[-1/2,-1/2]} + \prod_{i=1}^{N} n_{\mu_i}^{[1/2,1/2]}\right] + \mathrm{Re}\left[\prod_{i=1}^{N} n_{\mu_i}^{[-1/2,1/2]}\right], \tag{28}$$

where $n^{[\pm 1/2, \pm 1/2]} = (1, 0, 0, \pm 1)$ are the coordinates of the coherent states $|\frac{1}{2}, \pm\frac{1}{2}\rangle\langle\frac{1}{2}, \pm\frac{1}{2}|$ and $n^{[-1/2,1/2]} = 0, 1, -i, 0$ are the coordinates of the non-Hermitian operator $|\frac{1}{2}, -\frac{1}{2}\rangle\langle\frac{1}{2}, \frac{1}{2}|$.

While the complete characterization of the set of coordinates for which ρ is positive is difficult in any representation [18,21,23], our representation [Eq. (21)] allows one to solve this problem explicitely for $j = 1$. The set of all spin-1 states is characterized by 8 real parameters. The transformation of tensor $x_{\mu\nu}$ by rotation matrices under SU(2) operations allows one to diagonalize the 3×3 block x_{ab} $(1 \le a, b \le 3)$, and Eq. (5) imposes $\sum_{i=1}^{3} \mu_i = 1$ for the eigenvalues μ_i, leaving five real parameters μ_1, μ_2, $\mathbf{x} \equiv (x_{01}, x_{02}, x_{03})$. In this case, $\mathbf{x}$ coincides with $\mathbf{u}$ in the representation found in [38]. We therefore immediately obtain that up to two special cases of measure zero the set of all spin-1 states can be represented as a two–parameter family of ellipsoids in the space of vectors $\mathbf{x}$ (Eq. (21) in [38] with $\mathbf{u} = \mathbf{x}$ and $w_{ab} = x_{ab}$), thus providing a simple geometrical picture of all spin-1 states.

As a direct application of our formalism, we give a simple necessary and sufficient criterion for anticoherence of spin states. Spin states are said to be anticoherent to order t if $\langle(\mathbf{n}\cdot\mathbf{J})^k\rangle$ is independent on the unit vector $\mathbf{n}$ for any k with $0 \le k \le t$ [39]. Various characterizations have been given [40]. Very recently, the case of pure but not necessarily symmetric states was considered in [36,41]. The definition of matrices $S_{\mu_1\mu_2\ldots\mu_N}$ via (1) and (2) as a function of $\mathbf{J}$ makes them most convenient for the characterization of anticoherent states. One can show the following result:

Theorem 3: A spin-j state ρ, pure or mixed, is anticoherent to order t if and only if its spin-$(t/2)$ reduced density matrix is the maximally mixed state $\rho_0 = 1/(t+1)\mathbb{1}_{t+1}$.

The proof (see Supplemental Material for more detail [32]) relies on the calculation of $\langle\Pi^{(j)}(q)\rangle$ for an anticoherent state, using the Eqs. (3) and (4) and identifying terms up to order t with the expansion Eq. (27) of the maximally mixed state. For instance, spin-j anticoherent states to order 1 are characterized by $\langle S_{\mu 00\ldots 0}\rangle = \delta_{\mu 0}$ while anticoherent states to order 2 are characterized by $\langle S_{\mu\nu 00\ldots 0}\rangle = \mathrm{diag}(1, 1/3, 1/3, 1/3)$. From the characterization of anticoherence given by Theorem 3, one can easily obtain another characterization based on coefficients of the multipolar expansion of the density matrix. For a spin-j density operator ρ, the expansion reads

$$\rho = \sum_{k=0}^{2j} \sum_{q=-k}^{k} \rho_{kq} T_{kq}^{(j)} \tag{29}$$

with $\rho_{kq} = \mathrm{tr}(\rho T_{kq}^{(j)\dagger})$, where $T_{kq}^{(j)}$ are the irreducible tensor operators [20]

$$T_{kq}^{(j)} = \sqrt{\frac{2k+1}{2j+1}} \sum_{m,m'=-j}^{j} C_{jm,kq}^{jm'} |j, m'\rangle\langle j, m|, \tag{30}$$

and $C^{jm'}_{jm,kq}$ are Clebsch-Gordan coefficients. The following corollary of Theorem 3 can now be stated (see Supplemental Material [32] for a proof).

Corollary 1: A spin-j state ρ is anticoherent to order t if and only if $\rho_{kq} = 0, \forall k \leq t, \forall q\colon -k \leq q \leq k$.

Note that in [31] the current characterizations were obtained up to second order.

In summary, we have introduced a tensorial representation of spin states that leads to a natural generalization of the Bloch sphere representation to arbitrary spin j, based on Weinberg's covariant matrices [27]. We have found a convenient way of representing these matrices as projections of elements of the Pauli group into the symmetric subspace of $2j$ spins-$1/2$, proving that they form a tight frame. Our representation shares beautiful and essential properties with the one for spin-$1/2$ (or qubit), and provides additional insight for larger spins that we have used for a novel characterization of anticoherent spin states. We expect that the mathematical elegance of our representation will enable new insights in different fields of physics where spins are relevant.

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