Nonequivalence between absolute separability and positive partial transposition in the symmetric subspace

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The equivalence between absolutely separable states and absolutely positive partial transposed (PPT) states in general remains an open problem in quantum entanglement theory. In this work, we study an analogous question for symmetric multiqubit states. We show that symmetric absolutely PPT (SAPPT) states (symmetric states that remain PPT after any symmetry-preserving unitary evolution) are not always symmetric absolutely separable by providing explicit counterexamples. More precisely, we construct a family of entangled five-qubit SAPPT states. Similar counterexamples for larger odd numbers of qubits are identified.

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I. INTRODUCTION

The study of the properties of quantum states subject to unitary evolutions has been at the heart of quantum mechanics since its foundation. One of the most intriguing properties of a multipartite quantum state is entanglement, which can be created or destroyed by unitary evolutions. In a region surrounding the maximally mixed state, there exists a set of mixed states, called *absolutely separable* (AS), which remain separable under any global unitary transformation [1–3]. Characterizing this set is crucial as it defines the minimum conditions a quantum system must meet to attain a specific level of entanglement after a unitary transformation. The AS set in the bipartite scenario has been shown to be convex and compact [4], and its boundary has been determined for qubit-qudit systems [5]. A complete characterization of the AS set is still lacking, however, and remains an important open problem in quantum information science [6]. Absolute separability criteria can always be expressed in terms of mixed state eigenvalues, as these are the only quantities that are invariant under unitary transformations. Because of this, AS states are also called *separable from spectrum*.

The well-known positive partial transposition (PPT) criterion introduced by Peres [7] is a simple and powerful method, and therefore very useful, for testing entanglement, although it only provides a sufficient and not a necessary condition. By this criterion, a multipartite quantum state ρ is entangled in the $A|B$ bipartition if it is negative under partial transposition (NPT), i.e., ρ^{T_A} has at least one negative eigenvalue. This condition has been shown, in fact, to be necessary and sufficient for qubit-qubit and qubit-qutrit systems only [8–10]. This indicates the existence of entangled PPT states, also known as *bound entangled states*. Many explicit bound entangled states have been presented and studied [11–14]. In multipartite systems, the entanglement depends on the specific way the system is partitioned. A multipartite state is said to be (fully) separable if it can be written as a convex combination of multipartite product states [15]. Similarly, the PPT property depends on the bipartition considered. In this work, we are interested in states that are PPT for any bipartition, and we will call them just PPT states.

By analogy with the set of AS states, there is a set of absolutely PPT (APPT) states, i.e., states that remain PPT after any unitary evolution. In contrast with the AS set, the set of APPT states has been fully characterized through linear matrix inequalities [16]. Thus, because of the close connection between separable states and PPT states, a promising way to solve the *separability from spectrum* open problem would be to show that the APPT and AS sets are equivalent, as was undertaken in Ref. [17]. Although AS implies APPT according to the PPT criterion, the converse remains an open question. So far, the equivalence of AS and APPT has only been proved for qubit-qudit systems [18] and for various families of states [17]. To disprove the general equivalence between the sets, it suffices to provide a counterexample, that is, an entangled APPT state. However, finding a counterexample is a remarkably difficult task, as many known strong entanglement criteria fail to detect entanglement in APPT states [17]. Other criteria for states to be APPT and AS have been explored for higher-dimensional systems [19,20], as well as the extremal points of the AS and APPT sets for qutrit-qudit systems [21].

The absolute separability problem arises in many forms, notably for systems with continuous variables [22,23] and bosonic (symmetric) systems [20]. This work focuses on the latter case and more particularly on symmetric N-qubit systems. Their quantum states ρ and evolution are restricted to the symmetric subspace of dimension $N+1$ within the much larger Hilbert space of dimension 2^N. They satisfy $\rho = \pi\rho\pi'$ with π and π' any pair of permutation operators. The admissible unitary evolutions are also limited to $SU(N+1)$ transformations. A symmetric state ρ is said to be symmetric

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absolutely PPT (SAPPT) [20] if $U\rho U^\dagger$ is PPT for any $U \in \mathrm{SU}(N+1)$ [24]. Similarly, a state is symmetric absolutely separable (SAS) if the state remains separable after the action of any $U \in \mathrm{SU}(N+1)$. Due to the restriction on unitary operations that can be applied to the state, a SAS state is not necessarily AS [25]. A full parametrization of SAS states has been given for $N = 2$ [25], and additional SAS and SAPPT witnesses have been constructed [20,25–27]. However, the SAS and SAPPT sets remain to be fully characterized. A question similar to that of the nonsymmetric case then arises: Are SAPPT states always SAS? This is true for symmetric two- and three-qubit systems, which boil down to qubit-qubit and qubit-qutrit systems for which the PPT criterion is necessary and sufficient. But what about larger qubit systems? This question represents much more than a simplified version of the nonsymmetric case; it is of significant independent interest because of the many physical quantum systems constrained by permutation symmetry, such as Bose-Einstein condensates [28–30] and multiphoton systems [31]. In addition, it is of interest in the study of absolutely classical spin-j states, which are equivalent to SAS states [32]. More generally, entanglement in bosonic systems plays an important role as a resource in quantum metrology and quantum information [33]. In this work, we thus address the question of the general equivalence of SAPPT and SAS states. Our strategy is to search for counterexamples to the equivalence between the two sets. More specifically, we examine a subset of SAPPT states with a spectrum of a specific form and look for some entangled states in that subset.

The paper is organized as follows. Section II presents the full parametrization of the SAPPT states across a uniparametric spectrum. We then prove in Sec. III that for an odd number of qubits, some of these SAPPT states are entangled. Finally, we provide concluding remarks in Sec. IV. A summary table with all the acronyms used in this work, their meanings and definitions can be found in Appendix A.

II. UNIPARAMETRIC SPECTRUM OF SAPPT STATES

The symmetric sector of the Hilbert space of a N-qubit system, $\mathcal{H}^{\vee N}$, with $\mathcal{H}$ the single-qubit state space, is spanned by the Dicke states

$$\left|D_N^{(\alpha)}\right\rangle = \binom{N}{\alpha}^{-1/2} \sum_{\sigma} P_\sigma \big| \underbrace{00\ldots00}_{N-\alpha}\underbrace{11\ldots11}_{\alpha}\big\rangle \tag{1}$$

for $\alpha = 0, \ldots, N$, where the sum runs over all possible permutation operators P_σ of the N qubits. Let us now consider a symmetric state given by the mixture

$$\rho(p) = p\,\rho_0 + (1-p)\,|\psi_0\rangle\langle\psi_0| \tag{2}$$

of the maximally mixed state in the symmetric sector,

$$\rho_0 = \frac{1}{N+1}\sum_{m=0}^{N}\left|D_N^{(\alpha)}\right\rangle\left\langle D_N^{(\alpha)}\right| = \frac{\mathbb{1}_{N+1}}{N+1}, \tag{3}$$

with probability p, and a pure symmetric state $|\psi_0\rangle\langle\psi_0|$ with probability $1-p$, where $p \in [0, 1]$. The spectrum of $\rho(p)$ consists of two distinct eigenvalues: one nondegenerate eigenvalue $1 - \frac{Np}{N+1}$ and another N-fold degenerate eigenvalue $\frac{p}{N+1}$, i.e., $(1 - \frac{Np}{N+1}, \frac{p}{N+1}, \ldots, \frac{p}{N+1})$.

Now consider the bipartition of k and $N-k$ qubits $A|B$. We will denote the bipartitions of qubit systems that are in a symmetric state by $N_A|N_B = k|N-k$ with N_A the cardinality of the set A (and the same for B) since, for such states, the explicit specification of which qubits belong to which partition is irrelevant. Without loss of generality, we assume $k \leqslant \lfloor N/2 \rfloor$. Since ρ is a fully symmetric state, it has a support and an image in the tensor product of the symmetric sectors of subsystems A and B. Therefore, ρ is a convex combination of states in the bipartite Hilbert space $\mathcal{H}^{\vee k} \otimes \mathcal{H}^{\vee(N-k)}$. This Hilbert space with dimension $(k+1)\times(N-k+1)$ is spanned by the tensor products of the k- and $(N-k)$-qubit Dicke states, i.e., all the states $|D_k^{(\alpha)}\rangle\,|D_{N-k}^{(\beta)}\rangle \equiv |D_k^{(\alpha)}\rangle \otimes |D_{N-k}^{(\beta)}\rangle$ for $\alpha = 0, \ldots, k$ and $\beta = 0, \ldots, N-k$.

The partial transposed state ρ^{T_A} of (2) is the convex combination of $\rho_0^{T_A}$ and $\rho_{\psi_0}^{T_A} \equiv (|\psi_0\rangle\langle\psi_0|)^{T_A}$. The smallest eigenvalue of ρ^{T_A}, denoted by $\lambda_{\min}(\rho^{T_A})$, is lower bounded by

$$\lambda_{\min}\left(\rho^{T_A}\right) \geqslant \sigma(|\psi_0\rangle, p) \tag{4}$$

with

$$\sigma(|\psi_0\rangle, p) \equiv p\,\lambda_{\min}\left(\rho_0^{T_A}\right) + (1-p)\,\lambda_{\min}\left(\rho_{\psi_0}^{T_A}\right). \tag{5}$$

This lower bound follows from the result that the minimum eigenvalue of a sum of two Hermitian matrices, $A + B$, is greater than or equal to the sum of the minimum eigenvalues of each matrix A and B [34]. In particular, the equality holds when A and B have a common eigenvector for their lowest eigenvalue.

Let us first focus on the density matrix $\rho_0^{T_A}$. After some algebra (see Appendix B), we obtain that $\rho_0^{T_A}$ reads

$$\begin{aligned}\rho_0^{T_A} &= \frac{1}{N+1}\left(\sum_{\alpha=0}^{N}\left|D_N^{(\alpha)}\right\rangle\left\langle D_N^{(\alpha)}\right|\right)^{T_A} \\ &= \frac{1}{N+1}\sum_{\alpha=0}^{N}\sum_{\beta,\gamma=0}^{\alpha}\chi(\alpha,\beta)\,\chi(\alpha,\gamma) \\ &\quad\times\left|D_k^{(\alpha-\gamma)}\right\rangle\left|D_{N-k}^{(\beta)}\right\rangle\left\langle D_k^{(\alpha-\beta)}\right|\left\langle D_{N-k}^{(\gamma)}\right|,\end{aligned} \tag{6}$$

where

$$\chi(\alpha,\beta) \equiv \left[\frac{\binom{k}{\alpha-\beta}\binom{N-k}{\beta}}{\binom{N}{\alpha}}\right]^{1/2}. \tag{7}$$

We calculate in Appendix C the eigendecomposition of $\rho_0^{T_A}$. In particular, we prove that its minimum eigenvalue is given by

$$\lambda_{\min}\left(\rho_0^{T_A}\right) = \left[(N+1)\binom{N}{k}\right]^{-1}, \tag{8}$$

and that the states $|D_k^{(0)}\rangle\,|D_{N-k}^{(N-k)}\rangle$ and $|D_k^{(k)}\rangle\,|D_{N-k}^{(0)}\rangle$ are particular eigenvectors of $\rho_0^{T_A}$ with this eigenvalue.

Next, let us calculate $\lambda_{\min}(\rho_{\psi_0}^{T_A})$. We start with the Schmidt decomposition of $|\psi_0\rangle$ with respect to the bipartition $A|B$,

$$|\psi_0\rangle = \sum_{r=1}^{k+1} \sqrt{\Gamma_r}\,|\phi_r^A\rangle|\phi_r^B\rangle. \tag{9}$$

with $\Gamma_r \geqslant 0$, $\sum_r \Gamma_r = 1$ and, without loss of generality, we can assume that $\Gamma_r \geqslant \Gamma_{r+1}$. The set of states $\mathcal{A} = \{|\phi_r^A\rangle\}_{r=1}^{k+1}$ and $\mathcal{B} = \{|\phi_r^B\rangle\}_{r=1}^{k+1}$ form orthonormal bases of $\mathcal{H}^{\vee k}$ and $\mathcal{H}^{\vee(N-k)}$, respectively. The spectrum of $\rho_{\psi_0}^{T_A}$ can be calculated exactly [35,36], with $\lambda_{\min}(\rho_{\psi_0}^{T_A}) = -\sqrt{\Gamma_1\Gamma_2}$.

Using the previous results, we can now evaluate the lower bound $\sigma(|\psi_0\rangle, p)$ in Eq. (4), which gives

$$\sigma(|\psi_0\rangle, p) = \frac{p}{(N+1)\binom{N}{k}} - (1-p)\sqrt{\Gamma_1\Gamma_2}. \tag{10}$$

We can observe that this bound depends only on the Schmidt coefficients of $|\psi_0\rangle$. To deduce when the state is SAPPT, we are now interested in minimizing $\sigma(|\psi_0\rangle, p)$ on the unitary orbit of the state $\rho(p)$, i.e., on $\{U\rho(p)U^\dagger : U \in \mathrm{SU}(N+1)\}$. By Eq. (2),

$$U\rho(p)U^\dagger = p\rho_0 + (1-p)|\psi\rangle\langle\psi|, \tag{11}$$

where $|\psi\rangle = U|\psi_0\rangle$. This means that minimizing $\sigma(|\psi_0\rangle, p)$ on the unitary orbit of (2) reduces to minimizing $\sigma(|\psi\rangle, p)$ on the states $|\psi\rangle \in \mathcal{H}^{\vee N}$. Thus,

$$\min_{U\in\mathrm{SU}(N+1)} \lambda_{\min}\big(U\rho(p)^{T_A}U^\dagger\big) \geqslant \min_{|\psi\rangle\in\mathcal{H}^{\vee N}} \sigma(|\psi\rangle, p), \tag{12}$$

where the right-hand side is equal to

$$\frac{p}{(N+1)\binom{N}{k}} - \frac{1-p}{2} \tag{13}$$

and obtained for a state $|\psi\rangle$ with Schmidt coefficients $\Gamma_1 = \Gamma_2 = 1/2$ and the rest equal to zero. Thus, $\min_{U\in\mathrm{SU}(N+1)} \lambda_{\min}(\rho(p)^{T_A}) \geqslant 0$, and consequently $\rho(p)$ is SAPPT on the bipartition $k|N-k$, whenever Eq. (13) is greater or equal to zero. The strictest SAPPT condition is given by $k = \lfloor N/2 \rfloor$, where we can search for extremal values of p by setting Eq. (13) to zero. This leads to the sufficient condition of the following theorem, which describes a uniparametric set of SAPPT state spectra.

Theorem 1. Any symmetric N-qubit state ρ with a spectrum composed of $N+1$ nonzero eigenvalues of the form $(1-\frac{Np}{N+1}, \frac{p}{N+1}, \ldots, \frac{p}{N+1})$ is SAPPT if and only if $p \in [p_{\min}, 1]$ with

$$p_{\min} = \frac{1}{1 + 2\Big[(N+1)\binom{N}{\lfloor N/2 \rfloor}\Big]^{-1}}.$$

The necessary condition to be SAPPT for this family of states is proven by finding an NPT state for any p strictly smaller than $p_{\min}$. Consider the state $\rho(p)$ of Eq. (2) with $|\psi_0\rangle$ the N-qubit GHZ state $|\mathrm{GHZ}_N\rangle \equiv (|D_N^{(0)}\rangle - |D_N^{(N)}\rangle)/\sqrt{2}$. This state saturates the bound (12). Indeed, $\rho(p)^{T_A}$ for the bipartition $k|N-k$ has an eigenvector $(|D_k^{(0)}\rangle|D_{N-k}^{(N-k)}\rangle + |D_k^{(k)}\rangle|D_{N-k}^{(0)}\rangle)/\sqrt{2}$ with eigenvalue equal to Eq. (13), which is negative for $p < p_{\min}$, indicating that the state is NPT. We write in Table I the value of $p_{\min}$ for several numbers of qubits.

TABLE I. Particular values of the probability p defining the state $\rho(p)$ given by Eq. (2) with $|\psi_0\rangle = |\mathrm{GHZ}_N\rangle$, which has an eigenspectrum $(1-\frac{Np}{N+1}, \frac{p}{N+1}, \ldots, \frac{p}{N+1})$. First column: Number of qubits. Second column: Minimum value of p, $p_{\min}$ as it appears in Theorem 1, for $\rho(p)$ to be SAPPT. Third column: Maximum value $p_{\mathrm{ent}}^{W_N}$ such that the witnesses W_N given in Eqs. (14), (D1), and (D2) for $N = 5, 7$, and 9, respectively, detect that $\rho(p)$ is entangled. Fourth column: value p_{ent} below which the state $\rho(p)$ is found to be entangled using the method described in Ref. [37].

N	$p_{\min}$	$p_{\mathrm{ent}}^{W_N}$	p_{ent}
4	$\frac{15}{16}$	/	$\frac{15}{16}$
5	$\frac{30}{31} \approx 0.96774$	0.96862	0.96953
6	$\frac{70}{71}$	/	$\frac{70}{71}$
7	$\frac{140}{141} \approx 0.99291$	0.99302	0.99329
8	$\frac{315}{316}$	/	$\frac{315}{316}$
9	$\frac{630}{631} \approx 0.99842$	0.99845	0.99849
10	$\frac{1386}{1387}$	/	$\frac{1386}{1387}$

Following the same lines of reasoning, the sufficient condition of Theorem 1 can be generalized for symmetric states $|\psi\rangle \in \mathcal{H}_d^{\vee N}$ of N-qudit systems, with $\mathcal{H}_d$ the Hilbert space of an individual qudit and $D \equiv \dim \mathcal{H}_d^{\vee N} = \binom{N+d-1}{d-1}$. We observe by symbolic calculations that $\lambda_{\min}(\rho_0^{T_A}) = [D\binom{N}{k}]^{-1}$, for $d \leqslant 8$ and different values of $N \leqslant 15$. Since the strictest SAPPT condition is obtained again when $k = \lfloor N/2 \rfloor$, we can state the following conjecture.

Conjecture 1. Any symmetric N-qudit state ρ with a spectrum composed of $D = \binom{N+d-1}{d-1}$ nonzero eigenvalues of the form $(1-\frac{(D-1)p}{D}, \frac{p}{D}, \ldots, \frac{p}{D})$, with $p \in [p_{\min}, 1]$, where

$$p_{\min} = \frac{1}{1 + 2\Big[D\binom{N}{\lfloor N/2 \rfloor}\Big]^{-1}},$$

is SAPPT.

Theorem 1 is an improvement over a SAPPT criterion derived in Ref. [20] using invertible linear maps of operators.

III. ENTANGLED SAPPT STATES OF QUBITS

We are now ready to present a family of SAPPT states that are not SAS, i.e., entangled SAPPT states. Consider now the state (2) for five qubits with $|\psi_0\rangle = |\mathrm{GHZ}_5\rangle$. By Theorem 1, $\rho(p)$ is SAPPT for $p \geqslant p_{\min}$ and NPT for $p < p_{\min}$. However, we find that the state $\rho(p_{\min})$ is detected to be entangled by not having a two-copy PPT symmetric extension of the second party for the bipartition 1|4 (see Ref. [38] for more details). This can be checked using, for example, the QETLAB package [39]. This method, whenever the state is entangled, gives additionally an entanglement witness [38]. In our case, an entanglement witness for $\rho(p_{\min})$ is given in the Dicke-state basis of $\mathcal{H}^{\vee 5}$ by

$$W_5 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & c \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ c & 0 & 0 & 0 & 0 & a \end{pmatrix} \text{ with } \begin{cases} a = 0.0366656 \\ b = -0.134595 \\ c = -9.31947 \end{cases}. \tag{14}$$

One can easily check that indeed $\mathrm{Tr}(\rho(p_{\min})W_5) \approx -0.0085 < 0$. Moreover, one can check that W_5 is a proper entanglement witness for symmetric states by verifying that $W_5(\theta,\phi) \equiv \langle\theta,\phi|\,W_5\,|\theta,\phi\rangle \geqslant 0$ for all symmetric product states $|\theta,\phi\rangle = |\varphi\rangle^{\otimes N}$, with $|\varphi\rangle$ a single-qubit state parametrized as $\cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\phi}\,|1\rangle$. It suffices to check the positivity of the expectation value of W_5 on the set of pure product states $|\theta,\phi\rangle$ since any separable symmetric state ρ_{sep} can be written as a convex combination of them, i.e., $\rho_{\text{sep}} = \int P(\theta,\phi)\,|\theta,\phi\rangle\langle\theta,\phi|\,\mathrm{d}\Omega$ with $P(\theta,\phi) \geqslant 0$ and $\int P(\theta,\phi)\,\mathrm{d}\Omega = 1$ [40,41]. Thus, if the operator W_5 is such that $\langle\theta,\phi|\,W_5\,|\theta,\phi\rangle \geqslant 0$, then $\mathrm{Tr}(W_5\rho_{\text{sep}}) \geqslant 0$ for all separable states. The minimal value of $W_5(\theta,\phi)$ is reached at $(\theta,\phi) = (\pi/2, 0)$ and is approximately equal to 0.00276. Figure 1 (top panel) shows the finite values of $\ln[W_5(\theta,\phi)]$, illustrating the positivity of $W_5(\theta,\phi)$.

The operator W_5 given above also detects entanglement of $\rho(p)$ for values of p other than $p_{\min}$ (see Fig. 1, bottom panel). In fact, W_5 detects that $\rho(p)$ is entangled for $p \leqslant p_{\text{ent}}^{W_5} \approx 0.96862$, thus providing a uniparametric family of SAPPT bound entangled states $\rho(p)$ for $p \in [p_{\min}, p_{\text{ent}}^{W_5}]$. A larger family can be obtained using the reformulation of the separability problem as a truncated moment problem (see Ref. [37] for more details), which can be implemented as a semidefinite optimization. Using this method, with a precision of 10^{-5} in the determination of the parameter p, the state $\rho(p)$ was found to be entangled for $p_{\min} \leqslant p < p_{\text{ent}} = 0.96953$ and separable for $p_{\text{ent}} \leqslant p \leqslant 1$. Finally, we should mention that there are separable SAPPT states that are not SAS. An example is given by $\rho(p_{\min})$ with $|\psi_0\rangle$ a symmetric product state $|\theta,\phi\rangle$.

In a similar way, we searched for entangled SAPPT states for a number of qubits up to $N = 10$. For an odd number of qubits, the state (2) with $|\psi_0\rangle = |\mathrm{GHZ}_N\rangle$ is SAPPT and detected as entangled using QETLAB for values of p in the range $[p_{\min}, p_{\text{ent}}]$ (see Table I). The respective entanglement witnesses for $N = 7, 9$ are given in Appendix D. On the other hand, for even N, we find that the state $\rho(p)$ is always separable for any $p \in [p_{\min}, 1]$. So we could not find an example of an entangled SAPPT state for an even number of qubits.

IV. CONCLUSIONS

In this work, we established a sufficient condition for certain symmetric states of N-qudit systems to be SAPPT (symmetric absolutely PPT). For qubits, we showed that this condition is also necessary. In the course of this proof, we analytically determined the spectrum of $\rho_0^{T_A}$, where ρ_0 is the maximally mixed state in the symmetric subspace. Based on this, we proved the existence of entangled SAPPT states for an odd number of qubits from $N = 5$ by constructing explicit entanglement witnesses. These results resolve an open question concerning the equivalence between SAPPT and SAS states by showing that this equivalence does not hold in general, although it does apply to 2-qubit and 3-qubit systems. On the other hand, we have not yet identified entangled SAPPT states for an even number of qubits. It is important to note that in the nonsymmetric case, the entangled SAPPT states presented in this work do not refute the possibility of the equivalence between APPT and AS states. If APPT states were indeed equivalent to AS states, then the existence of entangled SAPPT states would provide a compelling example of how perfect indistinguishability among the constituents of multipartite systems, such as in bosonic systems, can profoundly affect their entanglement properties. We hope that our results will stimulate further exploration of this line of research in order to deepen our understanding of the interplay between symmetry and entanglement in quantum systems.

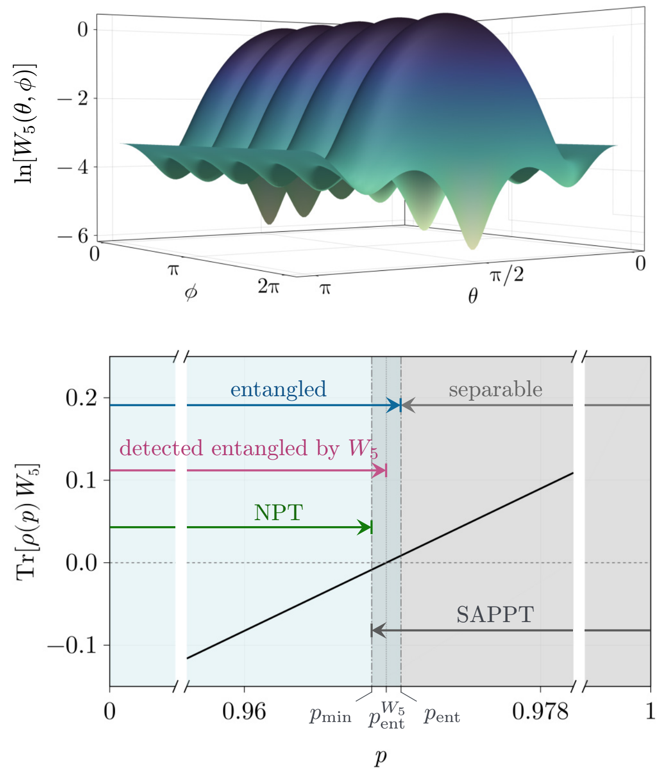

FIG. 1. Top panel: Logarithm of the expectation value of the entanglement witness W_5 over the pure symmetric product states, showing the positivity of $W_5(\theta,\phi)$. Bottom panel: Expectation value of the entanglement witness W_5 in $\rho(p)$ as a function of p (black oblique straight line). Entangled SAPPT states lie within the overlap between the blue (entangled states) and gray (SAPPT states) areas, ranging from $p = p_{\min} = 30/31 \approx 0.96774$ to $p_{\text{ent}} = 0.96953$. Those detected by the witness W_5 given in Eq. (14) lie between $p_{\min}$ and $p_{\text{ent}}^{W_5} \approx 0.96862 < p_{\text{ent}}$.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ACRONYMS

Table II contains all the acronyms used in this work.

APPENDIX B: DICKE STATES AS BIPARTITE STATES

For a given bipartition $k|N-k$ of N qubits, the symmetric Dicke states $|D_N^{(\alpha)}\rangle$ defined in Eq. (1) can be rewritten as

$$\begin{aligned}\left|D_N^{(\alpha)}\right\rangle &= \binom{N}{\alpha}^{-1/2}\sum_{r=0}^{\alpha}\left(\sum_{\sigma_r} P_{\sigma_r}\,|\underbrace{00\ldots00}_{N-\alpha-r}\underbrace{11\ldots11}_{k+\alpha+r-N}\rangle\right)\\ &\quad\otimes\left(\sum_{\lambda_r} P_{\lambda_r}\,|\underbrace{00\ldots00}_{r}\underbrace{11\ldots11}_{N-k-r}\rangle\right)\\ &= \sum_{r=0}^{\alpha}\left[\frac{\binom{k}{N-\alpha-r}\binom{N-k}{r}}{\binom{N}{\alpha}}\right]^{1/2}\left|D_k^{(k+\alpha+r-N)}\right\rangle\left|D_{N-k}^{(N-k-r)}\right\rangle,\end{aligned} \tag{B1}$$

where P_{σ_r} (respectively, P_{λ_r}) are permutation operators of k (respectively, $N-k$) qubits. After the change of variable from r to $\beta = N-k-r$, we get

$$\left|D_N^{(\alpha)}\right\rangle = \sum_{\beta=0}^{\alpha}\chi(\alpha,\beta)\left|D_k^{(\alpha-\beta)}\right\rangle\left|D_{N-k}^{(\beta)}\right\rangle,$$

with

$$\chi(\alpha,\beta) = \left[\frac{\binom{k}{\alpha-\beta}\binom{N-k}{\beta}}{\binom{N}{\alpha}}\right]^{1/2} = \left[\frac{\binom{\alpha}{\beta}\binom{N-\alpha}{k+\beta-\alpha}}{\binom{N}{k}}\right]^{1/2}$$

as defined in Eq. (7). It is precisely this last equation that we use to write $\rho_0^{T_A}$ as in Eq. (6).

APPENDIX C: EIGENSPECTRUM OF $\rho_0^{T_A}$

In this Appendix, we calculate the eigenspectrum of $\rho_0^{T_A} \equiv \frac{1}{N+1}\mathbb{1}_{N+1}^{T_A}$, with $\mathbb{1}_{N+1}$ the identity in the symmetric subspace of the N-qubit system and T_A the partial transposition performed on the first k qubits. To this aim, we consider a similar procedure as that used to calculate the spectra of angular momentum operators. First, we define the analog of the angular momentum operators $J_\pm$, J_z for the Dicke states, which we denote by $K_\pm$ and K_0. For all $\alpha = 0,\ldots,N$, we set

$$\begin{aligned}K_+\left|D_N^{(\alpha)}\right\rangle &\equiv \sqrt{(N-\alpha)(\alpha+1)}\left|D_N^{(\alpha+1)}\right\rangle,\\ K_-\left|D_N^{(\alpha)}\right\rangle &\equiv \sqrt{\alpha\,(N-\alpha+1)}\left|D_N^{(\alpha-1)}\right\rangle,\\ K_0\left|D_N^{(\alpha)}\right\rangle &\equiv \left(\frac{N}{2}-\alpha\right)\left|D_N^{(\alpha)}\right\rangle.\end{aligned} \tag{C1}$$

These operators verify the commutation relations $[K_+, K_-] = -2K_0$ and $[K_\pm, K_0] = \pm K_\pm$. In addition, $K_\pm^\dagger = K_\mp$. We now define new operators acting on $\mathcal{H}^{\vee k}\otimes\mathcal{H}^{\vee(N-k)}$ as follows:

$$M_\pm \equiv K_\mp^1 - K_\pm^2, \quad M_0 \equiv K_0^1 - K_0^2, \tag{C2}$$

where the superscript 1 (respectively, 2) refers to the subspace $\mathcal{H}^{\vee k}$ (respectively, $\mathcal{H}^{\vee(N-k)}$) on which the operator acts. By direct calculation, we obtain that

$$[M_0, M_\pm] = \pm M_\pm, \quad \left[M_\pm, \rho_0^{T_A}\right] = \left[M_0, \rho_0^{T_A}\right] = 0. \tag{C3}$$

By construction, a possible orthonormal basis of eigenvectors of M_0 are the Dicke product states $|D_k^{(\alpha)}\rangle\,|D_{N-k}^{(\beta)}\rangle$ ($\alpha = 0,\ldots,k;\beta = 0,\ldots,N-k$) with eigenvalue $k+\beta-\alpha-N/2$. These eigenvectors share the same eigenvalue as long as the difference $\beta-\alpha$ is the same. Hence, the eigenvalues of M_0 can be written as $\mu_m = m - N/2$ with $m = 0,\ldots,N$. A general eigenstate of M_0 with eigenvalue μ_m has the following general form:

$$|\phi_m\rangle = \sum_{r=\max(0,m+k-N)}^{\min(k,m)} d_r\left|D_k^{(k-r)}\right\rangle\left|D_{N-k}^{(m-r)}\right\rangle. \tag{C4}$$

The action of the ladder operators $M_\pm$ over these eigenstates fulfills the following proposition.

Proposition C1. The equation $M_-\,|\phi_m\rangle = 0$, with $|\phi_m\rangle$ an eigenstate of M_0 with respect to μ_m, admits a solution if and only if $m \leqslant k$, in which case it is unique (up to a normalization constant). Similarly, the equation $M_+\,|\phi_{N-m}\rangle = 0$, with $|\phi_{N-m}\rangle$ an eigenstate of M_0 with respect to μ_{N-m}, admits a solution if and only if $m \leqslant k$, in which case it is again unique up to a normalization constant.

Proof. Following Eq. (C4) and denoting the bounds of the summation as $r_1 = \max(0, m+k-N)$ and $r_2 = \min(k, m)$,

TABLE II. Table of acronyms used in this work, their meaning and definition. The subscript S refers to symmetric.

Acronym	Meaning	Definition
AS	Absolutely separable	$U\rho U^\dagger$ is separable for all unitary U
SAS	Symmetric absolutely separable	$U_S\rho_S U_S^\dagger$ is separable for all symmetry preserving unitary U_S
PPT	Positive partial transposed	$\rho^{T_A}\geqslant 0$
APPT	Absolutely positive partial transposed	$(U\rho U^\dagger)^{T_A}\geqslant 0$ for all unitary U
SAPPT	Symmetric absolutely positive partial transposed	$(U_S\rho_S U_S^\dagger)^{T_A}\geqslant 0$ for all symmetry preserving unitary U_S

the action of M_- on $|\phi_m\rangle$ gives

$$M_-\,|\phi_m\rangle = d_{r_1}\sqrt{r_1(k-r_1+1)}\,\big|D_k^{(k-r_1+1)}\big\rangle\,\big|D_{N-k}^{(m-r_1)}\big\rangle+\sum_{r=r_1}^{r_2-1}\Big(d_{r+1}\sqrt{(r+1)(k-r)} - d_r\sqrt{(m-r)(N-k-m+r+1)}\Big)\big|D_k^{(k-r)}\big\rangle\big|D_{N-k}^{(m-r-1)}\big\rangle - d_{r_2}\sqrt{(m-r_2)(N-k-m+r_2+1)}\,\big|D_k^{(k-r_2)}\big\rangle\big|D_{N-k}^{(m-r_2-1)}\big\rangle. \tag{C5}$$

To fulfill $M_-\,|\phi_m\rangle=0$, the coefficients d_r must obey

$$d_{r_1}=0,\quad \text{if } r_1\neq 0,\qquad d_{r_2}=0,\quad \text{if } r_2\neq m, \tag{C6}$$

and the recursive relation, $\forall r=r_1,\dots,r_2-1$,

$$d_{r+1}=\left[\frac{(m-r)(N-k-m+r+1)}{(r+1)(k-r)}\right]^{1/2} d_r. \tag{C7}$$

If $r_1\neq 0$ or $r_2\neq m$, then all d_r must be zero by the recursive relation. This is the case when $m>k$. Conversely, if $m\leqslant k$, $r_1=0$, $r_2=m$, and the recursive relation yields $\forall r=1,\dots,m$, then

$$d_r=\left[\frac{\binom{k-r}{m-r}\binom{N-k-m+r}{r}}{\binom{k}{m}}\right]^{1/2} d_0. \tag{C8}$$

This defines a unique state $|\phi_m\rangle$ up to a normalization constant. Using an alternative version of Vandermonde convolution in combinatorics [42],

$$\binom{\alpha+\beta}{\gamma}=\sum_{\kappa=0}^{\gamma}\binom{\alpha-\kappa}{\gamma-\kappa}\binom{\beta+\kappa-1}{\kappa}, \tag{C9}$$

we easily obtain that the state $|\phi_m\rangle$ gets normalized with

$$d_0=\binom{k}{m}^{1/2}\binom{N-m+1}{m}^{-1/2}. \tag{C10}$$

For M_+, a similar derivation holds. ■

We now describe the common eigenbasis of $\rho_0^{T_A}$ and M_0 as a Theorem.

Theorem C1. The operators $\rho_0^{T_A}$ and M_0, defined in Eqs. (6) and (C2), form a complete set of commuting observables (CSCO) whose common eigenvectors generate a basis of the Hilbert space $\mathcal{H}^{\vee k}\otimes\mathcal{H}^{\vee N-k}$ with $k\leqslant N/2$. The normalized eigenvectors $|n,m\rangle$ are identified by two quantum numbers $n=0,\dots,k$ and $m=n,\dots N-n$, and satisfy the eigenvalue equations

$$\rho_0^{T_A}\,|n,m\rangle=\lambda_n\,|n,m\rangle,\qquad M_0\,|n,m\rangle=\mu_m\,|n,m\rangle, \tag{C11}$$

with

$$\lambda_n=\frac{1}{N+1}\frac{\binom{N+1}{n}}{\binom{N}{k}},\qquad \mu_m=m-\frac{N}{2}. \tag{C12}$$

The eigenvectors $|n,n\rangle$ and $|n,N-n\rangle$ explicitly read

$$|n,n\rangle=\mathcal{N}\sum_{r=0}^{n}c_r^{k,n}\,\big|D_k^{(k-r)}\big\rangle\,\big|D_{N-k}^{(n-r)}\big\rangle, \tag{C13}$$

$$|n,N-n\rangle=\mathcal{N}\sum_{r=0}^{n}c_r^{N-k,n}\,\big|D_k^{(n-r)}\big\rangle\,\big|D_{N-k}^{(N-k-r)}\big\rangle, \tag{C14}$$

with $\mathcal{N}=\binom{N-n+1}{n}^{-1/2}$ and

$$c_r^{k,n}=\left[\binom{k-r}{n-r}\binom{N-k-n+r}{r}\right]^{1/2}. \tag{C15}$$

For $n<m<N-n$, we have

$$|n,m\rangle\propto(M_+)^{m-n}\,|n,n\rangle,\qquad |n,N-m\rangle\propto(M_-)^{m-n}\,|n,N-n\rangle. \tag{C16}$$

Proof. For each $n=0,\dots,k$, let us consider the unique (normalized) eigenstate $|\phi_n\rangle$ of M_0 with respect to μ_n that fulfills $M_-\,|\phi_n\rangle=0$. This state is given by Eqs. (C4), (C8), and (C10) with $m=n$. The action of $\rho_0^{T_A}$ on it reads

$$(N+1)\rho_0^{T_A}\,|\phi_n\rangle=\sum_{s,r=0}^{n}d_r\,\chi(k+n-r-s,n-s)\times\chi(k+n-r-s,n-r)\,\big|D_k^{(k-s)}\big\rangle\big|D_{N-k}^{(n-s)}\big\rangle. \tag{C17}$$

Using standard binomial identities [42], we get that

$$\chi(k+n-r-s,n-s)\,\chi(k+n-r-s,n-r)$$
$$= \frac{\binom{k+n-r-s}{n-r}\binom{N-k-n+r+s}{r}}{\binom{N}{k}}\left[\frac{\binom{k-s}{n-s}\binom{N-k-n+s}{s}}{\binom{k-r}{n-r}\binom{N-k-n+r}{r}}\right]^{1/2}$$
$$= \frac{\binom{k+n-r-s}{n-r}\binom{N-k-n+r+s}{r}}{\binom{N}{k}}\,\frac{d_s}{d_r}. \tag{C18}$$

Thus,

$$\sum_{r=0}^{n} d_r\,\chi(k+n-r-s,n-s)\,\chi(k+n-r-s,n-r)$$
$$= \frac{d_s}{\binom{N}{k}}\sum_{r=0}^{n}\binom{k+n-r-s}{n-r}\binom{N-k-n+r+s}{r}$$
$$= \frac{\binom{N+1}{n}}{\binom{N}{k}}d_s, \tag{C19}$$

where we used Eq. (C9) in the last equality. Inserting Eq. (C19) in Eq. (C17), we obtain that $\rho_0^{T_A}\,|\phi_n\rangle = \lambda_n\,|\phi_n\rangle$. Hence the unique eigenstate $|\phi_n\rangle$ of M_0 with respect to μ_n that fulfills in addition $M_-\,|\phi_n\rangle = 0$ is also an eigenstate of $\rho_0^{T_A}$ with respect to λ_n. For $n = 0,\ldots,k$, we can denote accordingly these states by $|\lambda_n,\mu_n\rangle$ or just $|n,n\rangle$ for short. They are given by Eqs. (C13) and (C15). In a very similar way, we obtain that the unique eigenstate $|\phi_{N-n}\rangle$ of M_0 with respect to μ_{N-n} that fulfills in addition $M_+\,|\phi_{N-n}\rangle = 0$ is also an eigenstate of $\rho_0^{T_A}$ with respect to λ_n. They can be denoted accordingly $|n,N-n\rangle$.

Because of the commutation relations (C3), the state $M_+\,|n,n\rangle$ is either zero or a common eigenstate of $\rho_0^{T_A}$ and M_0 with respective eigenvalues λ_n and μ_{n+1}. In the latter case, we can denote it by $|n,n+1\rangle$ after normalization. Again, $M_+\,|n,n+1\rangle$ is either zero or generates a new common eigenstate $|n,n+2\rangle$. We can iterate this procedure a finite number of steps q (since our Hilbert space is finite) until we get a state $|n,n+q\rangle$ that fulfills $M_+\,|n,n+q\rangle = 0$. In this case we know from above that this state must identify to $|N-(n+q),n+q\rangle$, and consequently $q = N-2n$. In this way, we obtain $N+1-2n$ common eigenstates $|n,m\rangle$ ($m = n,\ldots,N-n$) for each $n = 0,\ldots,k$. The total number of these eigenstates amounts to

$$\sum_{n=0}^{k}(N+1-2n) = (k+1)(N-k+1), \tag{C20}$$

which matches the dimension of the Hilbert space. ■

Let us give an example. For $N = 5$ qubits and the bipartition of $k = 2$ and $N-k = 3$ qubits, we present in Fig. 2 the allowed quantum numbers defining $(k+1)(N-k+1) = 12$ eigenstates. We note that the eigenspectrum of $\rho_0^{T_A}$ is highly degenerate. The most-left (most-right) points are associated to the states that are eliminated by M_- (M_+).

Finally we may note that $0 < \lambda_n < 2/(N+1)$, for all k and N.

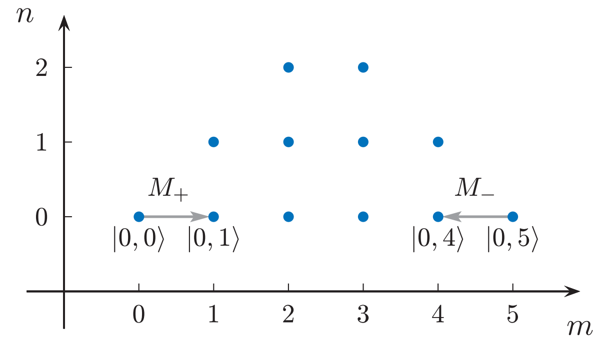

FIG. 2. Schematic representation of the common eigenvectors $|n,m\rangle$ of $\rho_0^{T_A}$ and M_0 with respect to the quantum numbers n and m for $N = 5$ and $k = 2$. Each dot represents a common eigenvector and the dot ensemble illustrates the allowed pairs of quantum numbers (n,m).

APPENDIX D: ENTANGLEMENT WITNESSES FOR $N > 5$

For $N = 7$, the state ρ given by (2) with $|\psi_0\rangle = |\mathrm{GHZ}_7\rangle$ for $p = p_{\min} = 140/141$ is detected to be entangled by not having two-copy PPT symmetric extension of the second party for the bipartitions 1|6. An entanglement witness in the Dicke basis is given by

$$W_7 = \begin{pmatrix} a&0&0&0&0&0&0&d\\ 0&b&0&0&0&0&0&0\\ 0&0&c&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&c&0&0\\ 0&0&0&0&0&0&b&0\\ d&0&0&0&0&0&0&a \end{pmatrix}$$
$$\text{with}\ \begin{cases} a = 0.00197514\\ b = 0.0643064\\ c = -0.189017\\ d = -31.2405 \end{cases}, \tag{D1}$$

from which we get $\mathrm{Tr}(\rho W_7) \approx -0.0038 < 0$ and $\min W_7(\theta,\phi) \approx 0.001975$ at $(\theta,\phi) = (0,0)$. Taking into account the results of Theorem 1, W_7 detects that $\rho(p)$ is an entangled SAPPT state for $p_{\min} \leqslant p \leqslant p_{\mathrm{ent}}^{W_7} \approx 0.9930$.

Similarly, for $N = 9$, the state ρ given by (2) with $|\psi_0\rangle = |\mathrm{GHZ}_9\rangle$ for $p_{\min} = 630/631$ is detected to be entangled by not having a two-copy PPT symmetric extension of the second party for the bipartitions 4|5. An entanglement witness in the Dicke basis is given by

$$W_9 = \begin{pmatrix} a&0&0&0&0&0&0&0&0&e\\ 0&b&0&0&0&0&0&0&0&0\\ 0&0&c&0&0&0&0&0&0&0\\ 0&0&0&d&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&d&0&0&0\\ 0&0&0&0&0&0&0&c&0&0\\ 0&0&0&0&0&0&0&0&b&0\\ e&0&0&0&0&0&0&0&0&a \end{pmatrix} \tag{D2}$$

with

$$\begin{cases} a = 0.00235791 \\ b = -0.013747 \\ c = 0.0621661 \\ d = -0.1636915 \\ e = -114.305 \end{cases}.$$

We have $\mathrm{Tr}(\rho W_9) \approx -0.004 < 0$ and $\min W_9(\theta, \phi) \approx 0.0002234$ at $(\theta, \phi) = (0.381, 0)$. In the range $p_{\min} \leqslant p \leqslant 1$, i.e., when the state is SAPPT, W_9 detects that ρ is entangled for $p_{\min} \leqslant p \leqslant p_{\mathrm{ent}}^{W_9} \approx 0.99845$.

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