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Radiation pressure on a two-level atom: an exact analytical approach

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The mechanical action of light on atoms is currently a tool used ubiquitously in cold atom physics. In the semiclassical regime, where atomic motion is treated classically, the computation of the mean force acting on a two-level atom requires numerical approaches in the most general case. Here we show that this problem can be tackled in a purely analytical way. We provide an analytical yet simple expression of the mean force that holds in the most general case, where the atom is simultaneously exposed to an arbitrary number of lasers with arbitrary intensities, wave vectors, and phases. This yields a novel tool for engineering the mechanical action of light on single atoms.

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1. INTRODUCTION

With the advent of lasers, the mechanical action of light has become an extraordinary tool for controlling the motion of atoms. The first evidence of this control with laser light was demonstrated in the early 1970s with the deflection of an atomic beam by resonant laser radiation pressure [1,2]. One of the most remarkable achievements was made a decade later when the first cold atomic cloud in a magneto-optical trap was observed [3]. These initial experiments gave birth to a vast array of cold atom physics experiments, with each more spectacular than the last. Regimes (for example, Bose–Einstein condensation [4]) that were thought to forever remain in the realm of Gedankenexperiments became reality in labs.

As long as atomic motion can be treated classically, i.e., in regimes where the atomic wave packets are sufficiently localized in space, the resonant laser radiation acts mechanically as a force on the atomic center-of-mass. In a standard two-level approximation and modeling the laser electromagnetic field as a plane wave, the mean force **F** exerted by a single laser, averaged over its optical period, reaches a stationary regime shortly after establishment of the laser action, where it takes the well-known simple expression [5]

$$\mathbf{F} = \frac{\Gamma}{2}\frac{s}{1+s}\hbar\mathbf{k}. \tag{1}$$

Here, Γ is the rate of spontaneous decay from the upper level of the transition, $\hbar\mathbf{k}$ is the laser photon momentum, and $s = (|\Omega|^2/2)/(\delta^2 + \Gamma^2/4)$ is the saturation parameter, where Ω is the Rabi frequency, and $\delta = \omega - \omega_0$ is the detuning between the laser and the atomic transition angular frequencies, ω and ω_0, respectively. The maximal force the laser can exert on the atom is $(\Gamma/2)\hbar k$.

The question naturally arises of how Eq. (1) generalizes when several lasers of arbitrary intensities, wave vectors, and phases act simultaneously on the atom. Surprisingly, to date no general exact analytical expression of the resulting force can be found in the scientific literature, and one is often reduced to using numerical approaches [6–13]. In the low-intensity regime, a generalized version of Eq. (1) provides an approximation of the incoherent action of each laser field. Each individual laser j ($j = 1, \ldots, N$ with N the total number of lasers) is characterized by an individual detuning δ_j, a photon momentum $\hbar\mathbf{k}_j$, a Rabi frequency Ω_j, and an individual saturation parameter $s_j = (|\Omega_j|^2/2)/(\delta_j^2 + \Gamma^2/4)$. When all s_j are much smaller than 1, the mean resulting force exerted incoherently by all lasers on the atom can be approximated by $\mathbf{F} = \Sigma_j \mathbf{F}_j$, where

$$\mathbf{F}_j = \frac{\Gamma}{2}\frac{s_j}{1+s_{\text{eff}}}\hbar\mathbf{k}_j \tag{2}$$

is the mean force exerted by each individual laser j, and $s_{\text{eff}} = \Sigma_j s_j$ [14]. For larger values of s_j, Eq. (2) loses validity, as it is derived in a rate-equation approximation [14]. It also neglects any coherent action of the lasers. Coherence effects can lead to huge forces that vastly exceed the maximal value of $(\Gamma/2)\hbar k_j$ per laser, as is observed with the stimulated bichromatic force [6]. An exact expression of the force, including coherent effects, that holds at high intensity can be found in the particular case of a pair of counterpropagating lasers of the same intensity [15].

In this paper, we solve analytically the most general case and provide an expression for the force exerted by an arbitrary number of lasers with arbitrary intensities, phases, detunings, and directions acting on the same individual two-level atom. We show how to express this force in strict generality in the form of Eq. (2), even including coherent effects, with a generalized definition of the saturation parameter s_j. We thus provide a unified formalism that holds in any configuration of lasers and enables the engineering of the mechanical action of light on individual atoms.

The paper is organized as follows: In Section 2, the exact expression of the radiation pressure force in the most general configuration is calculated. In Section 3, some specific regimes are investigated, where some interesting simplifications hold. Finally, we draw conclusions in Section 4.

2. GENERAL AND EXACT EXPRESSION OF THE RADIATION PRESSURE FORCE

We consider a two-level atom with levels $|e\rangle$ and $|g\rangle$ of energy E_e and E_g, respectively ($E_e > E_g$). We denote the atomic transition angular frequency $(E_e - E_g)/\hbar$ by ω_{eg}. The atom interacts with a classical electromagnetic field $\mathbf{E}(\mathbf{r}, t)$ resulting from the superposition of N arbitrary plane waves: $\mathbf{E}(\mathbf{r}, t) = \Sigma_j \mathbf{E}_j(\mathbf{r}, t)$, with $\mathbf{E}_j(\mathbf{r}, t) = (\mathbf{E}_j/2)e^{i(\omega_j t - \mathbf{k}_j\cdot\mathbf{r}+\varphi_j)} + \text{c.c.}$ Here, ω_j, $\mathbf{k}_j$ and φ_j are the angular frequency, the wave vector, and the phase of the jth plane wave, respectively, and $\mathbf{E}_j \equiv E_j\boldsymbol{\epsilon}_j$, with $E_j > 0$ and $\boldsymbol{\epsilon}_j$ the normalized polarization vector of the corresponding wave. The quasi-resonance condition is fulfilled for each plane wave: $|\delta_j| \ll \omega_{eg}$, $\forall j$, where $\delta_j = \omega_j - \omega_{eg}$ is the detuning. We define a weighted mean frequency of the plane waves by $\overline{\omega} = \Sigma_j \kappa_j \omega_j$, with $\{\kappa_j\}$ an *a priori* arbitrary set of weighting factors ($\kappa_j \geq 0$ and $\Sigma_j \kappa_j = 1$).

In the electric-dipole approximation and considering spontaneous emission in the master equation approach [16], the atomic density operator $\hat{\rho}$ obeys

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}[\hat{H}(t), \hat{\rho}(t)] + \mathcal{D}(\hat{\rho}(t)) \tag{3}$$

with $\hat{H}(t) = \hbar\omega_e|e\rangle\langle e| + \hbar\omega_g|g\rangle\langle g| - \hat{\mathbf{D}}\cdot\mathbf{E}(\mathbf{r}, t)$ and $\mathcal{D}(\hat{\rho}) = (\Gamma/2)([\hat{\sigma}_-, \hat{\rho}\hat{\sigma}_+] + [\hat{\sigma}_-\hat{\rho}, \hat{\sigma}_+])$, where $\hat{\mathbf{D}}$ is the atomic electric dipole operator, $\mathbf{r}$ is the atom position in the electric field, Γ is the spontaneous de-excitation rate of the excited state $|e\rangle$, and $\hat{\sigma}_- \equiv |g\rangle\langle e|$ and $\hat{\sigma}_+ \equiv |e\rangle\langle g|$ are the atomic lowering and raising operators, respectively.

The hermiticity and the unit trace of the density operator make all four matrix elements ρ_{ee}, ρ_{eg}, ρ_{ge}, and ρ_{gg}, with $\rho_{kl} = \langle k|\hat{\rho}|l\rangle$ for $k, l = e, g$ dependent variables. We consider here the vector of real independent variables $\mathbf{x} = (u, v, w)^T$, with $u = \text{Re}(\tilde{\rho}_{ge})$, $v = \text{Im}(\tilde{\rho}_{ge})$, and $w = (\rho_{ee} - \rho_{gg})/2 = \rho_{ee} - 1/2$, where $\tilde{\rho}_{ge} = \rho_{ge}e^{-i\overline{\omega}t}$. In the rotating wave approximation (RWA), the time evolution of $\mathbf{x}$ resulting from Eq. (3) obeys

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b} \tag{4}$$

with $\mathbf{b} = (0, 0, -\Gamma/2)^T$ and

$$A(t) = \begin{pmatrix} -\Gamma/2 & \overline{\delta} & \text{Im}(\Omega(t)) \\ -\overline{\delta} & -\Gamma/2 & -\text{Re}(\Omega(t)) \\ -\text{Im}(\Omega(t)) & \text{Re}(\Omega(t)) & -\Gamma \end{pmatrix}, \tag{5}$$

where $\overline{\delta} = \overline{\omega} - \omega_{eg}$ and $\Omega(t) = \Sigma_j \Omega_j e^{i(\omega_j - \overline{\omega})t}$, with Ω_j ($j = 1, \ldots, N$) the complex Rabi frequencies

$$\Omega_j = -\mathbf{D}_{ge}\cdot\mathbf{E}_j e^{i(-\mathbf{k}_j\cdot\mathbf{r}+\varphi_j)}/\hbar \equiv \Omega_{\text{R},j}e^{i\phi_j}, \tag{6}$$

where $\mathbf{D}_{ge} = \langle g|\hat{\mathbf{D}}|e\rangle$, $\Omega_{\text{R},j} > 0$ denotes the modulus of Ω_j, and ϕ_j its phase. Without loss of generality, the global phases of the atomic states $|e\rangle$ and $|g\rangle$ can always be chosen so as to have one Rabi frequency real and positive. This is usually considered in all studies where a single plane wave interacts with the atom. However, with N arbitrary plane waves, we cannot assume without loss of generality that all Rabi frequencies are real, and their phases cannot be ignored. In the quasi-resonance condition, the RWA is fully justified as long as $\Omega_{\text{R},j}/\omega_j \ll 1$, $\forall j$ [17].

Equation (4) constitutes the so-called optical Bloch equations (OBEs) adapted to the present studied case. Here, because of the time dependence of $A(t)$, the solution cannot be expressed analytically, and the equation must be integrated numerically. However, within an arbitrary accuracy, we can always assume that the N frequency differences $\omega_j - \overline{\omega}$ are commensurable, i.e., that all ratios $(\omega_j - \overline{\omega})/(\omega_l - \overline{\omega})$, $\forall j, l{:}\omega_l \neq \overline{\omega}$, are rational numbers. Within this assumption, $\Omega(t)$ and $A(t)$ are periodic in time with a period $T_c = 2\pi/\omega_c$, where $\omega_c = (\text{LCM}[(\omega_j - \overline{\omega})^{-1}, \forall j{:}\omega_j \neq \overline{\omega}])^{-1}$, with LCM denoting the least common multiple conventionally taken as positive. It also follows that the numbers $(\omega_j - \overline{\omega})/\omega_c$, hereafter denoted by m_j, are integer, $\forall j$ [18]. In the particular case where all frequencies ω_j are identical, $A(t)$ is constant in time, or, equivalently, periodic with an arbitrary value of $\omega_c \neq 0$, and all integer numbers m_j vanish.

Within the commensurability assumption where $A(t)$ is T_c-periodic and given initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, the OBEs admit the unique solution (Floquet's theorem; see, e.g., Ref. [19])

$$\mathbf{x}(t) = P_I(t)e^{R(t-t_0)}(\mathbf{x}_0 - \mathbf{x}_p(t_0)) + \mathbf{x}_p(t), \tag{7}$$

where R is a logarithm of the OBE monodromy matrix divided by T_c [20], $P_I(t)$ is an invertible T_c-periodic matrix equal to $X_I(t)e^{-R(t-t_0)}$ for $t \in [t_0, t_0 + T_c]$, with $X_I(t)$ the matriciant of the OBEs [20], and $\mathbf{x}_p(t)$ is an arbitrary particular solution of the OBEs. The real parts of the eigenvalues of the matrix R, the so-called Floquet exponents, belong to the interval $[\int_0^{T_c}\lambda_{\min}(t)dt/T_c, \int_0^{T_c}\lambda_{\max}(t)\text{d}t/T_c]$ with $\lambda_{\min}(t)$ and $\lambda_{\max}(t)$ the minimal and maximal eigenvalues of the matrix $[A(t) + A^\dagger(t)]/2$, respectively [21]. Here, this matrix reads $\text{diag}(-\Gamma/2, -\Gamma/2, -\Gamma)$, and the real parts of the Floquet exponents are thus necessarily comprised between $-\Gamma$ and $-\Gamma/2$, hence, strictly negative. This implies, first, that the matrix $e^{R(t-t_0)}$ tends to zero with a characteristic time not shorter than Γ^{-1} and not longer than $2\Gamma^{-1}$. At long times ($t - t_0 \gg 2\Gamma^{-1}$) $\mathbf{x}(t) \simeq \mathbf{x}_p(t)$. Second, the OBEs are ensured to admit a unique T_c-periodic solution [19] that the particular solution $\mathbf{x}_p(t)$ can be set to. This certifies that at long times the solution of the OBEs is necessarily periodic (periodic regime).

The unique T_c-periodic solution of the OBEs can be expressed using the Fourier expansion

$$\mathbf{x}(t) = \sum_{n=-\infty}^{+\infty} \mathbf{x}_n e^{\mathrm{i}n\,\omega_c t}, \tag{8}$$

with $\mathbf{x}_n \equiv (u_n, v_n, w_n)^T$ the Fourier components of $\mathbf{x}(t)$. Since $\mathbf{x}(t)$ is real, $\mathbf{x}_{-n} = \mathbf{x}_n^*$, and since it is continuous and differentiable, $\Sigma_n|\mathbf{x}_n|^2 < \infty$. Inserting Eq. (8) into the OBEs yields an infinite system of equations connecting all u_n, v_n and w_n components. The system can be rearranged so as to express all u_n and v_n as a function of the w_n components. Proceeding in this way yields, $\forall\, n$,

$$u_n = -i\left(\tau_n^+ \sum_{j=1}^{N} \Omega_j w_{n-m_j} - \tau_n^- \sum_{j=1}^{N} \Omega_j^* w_{n+m_j}\right),$$

$$v_n = -\left(\tau_n^+ \sum_{j=1}^{N} \Omega_j w_{n-m_j} + \tau_n^- \sum_{j=1}^{N} \Omega_j^* w_{n+m_j}\right), \tag{9}$$

with $\tau_n^{\pm} = 1/[\Gamma + 2i(n\omega_c \pm \bar{\delta})]$ and

$$w_n + \sum_{m\in M_0} \mathcal{W}_{n,m} w_{n+m} = \frac{-1}{2(1+\tilde{s})}\delta_{n,0}. \tag{10}$$

Here, $\delta_{n,0}$ denotes the Kronecker symbol, M_0 is the set of all distinct nonzero integers $m_{lj} \equiv m_l - m_j$ $(j, l = 1, \ldots, N)$, $\mathcal{W}_{n,m} = \beta_{n,m}/(\alpha_n + \beta_{n,0})$ $(m \in \mathbb{Z})$, with $\alpha_n = \Gamma + \mathrm{i}n\,\omega_c$ and $\beta_{n,m} = \Sigma_{j,l:m_{lj}=m}\Omega_j\Omega_l^*(\tau_{n+m_l}^+ + \tau_{n-m_j}^-)$, and $\tilde{s} = \Sigma_j \tilde{s}_j$, with

$$\tilde{s}_j = \mathrm{Re}\left[\frac{\Omega_j}{\Gamma/2 - i\delta_j} \sum_{\substack{l=1 \\ l:\delta_l=\delta_j}}^{N} \frac{\Omega_l^*}{\Gamma}\right]. \tag{11}$$

We have $\tau_{-n}^{\pm} = \tau_n^{\mp *}$, $\alpha_{-n} = \alpha_n^*$, $\beta_{-n,-m} = \beta_{n,m}^*$, and $\mathcal{W}_{-n,-m} = \mathcal{W}_{n,m}^*$.

If we define the vector of all w_n components for n ranging from $-\infty$ to $+\infty$, $\mathbf{w} = (\ldots, w_{-1}, w_0, w_1, \ldots)^T$, and the infinite matrix W of elements $W_{n,n'} = \Sigma_{m\in M_0}\mathcal{W}_{n,m}\delta_{n',n+m}$ $(n, n'$ ranging from $-\infty$ to $+\infty)$, Eq. (10) yields the complex inhomogeneous infinite system of equations

$$(I + W)\mathbf{w} = \mathbf{c}, \tag{12}$$

with I the infinite identity matrix and $\mathbf{c}$ the infinite vector of elements $c_n = -\delta_{n,0}/[2(1+\tilde{s})]$. W is an infinite centrohermitian band-diagonal matrix with as many bands as the cardinality of M_0. Its main diagonal is zero. Since the OBEs admit a unique T_c-periodic solution, the infinite system admits a unique solution $\mathbf{w}$ with the property $\Sigma_n|w_n|^2 < \infty$. In these conditions and observing that $\Sigma_n|c_n|^2 < \infty$ and that the series $\Sigma_{n,n'} W_{n,n'} = \Sigma_n\Sigma_{m\in M_0}\mathcal{W}_{n,m}$ is absolutely convergent whatever the values of the Rabi frequencies Ω_j, the detunings δ_j, and the de-excitation rate Γ, the infinite system [Eq. (12)] can be solved via finite larger and larger truncations, whose solutions are ensured to converge in all cases to the unique sought solution $\mathbf{w}$ [22]. This solution is necessarily such that $w_0 \neq 0$; otherwise, all other coefficients w_n would solve a homogeneous system of equations and vanish, in which case the equation for $n = 0$ could not be satisfied. This allows us to define the ratios $q_n = w_n/w_0$, $\forall\, n$. In particular $q_0 = 1$, and the reality condition yields $q_{-n} = q_n^*$. We have (Cramer's rule)

$$q_n = \lim_{k\to\infty} \frac{\Delta_k^{(n)}}{\Delta_k^{(0)}}, \tag{13}$$

with $\Delta_k^{(n)}$ the determinant of the $I + W$ matrix truncated to the lines and columns $-k, \ldots, k$ and where the n-indexed column is replaced by the vector $\mathbf{c}$, correspondingly truncated. As argued above, the limit is ensured to exist in all cases [23]. Inserting $w_m = q_m w_0$ in Eq. (10) for $n = 0$ allows for expressing w_0 in the form

$$w_0 = -\frac{1}{2}\frac{1}{1+s_{\mathrm{eff}}}, \tag{14}$$

where $s_{\mathrm{eff}} = \Sigma_j s_j$, with newly defined parameters s_j as

$$s_j = \mathrm{Re}\left[\frac{\Omega_j}{\Gamma/2 - i\delta_j} \sum_{l=1}^{N} \frac{\Omega_l^*}{\Gamma} q_{m_{lj}}\right]. \tag{15}$$

According to the Ehrenfest theorem, the mean power absorbed by the atom from the jth plane wave, $P_j(t) \equiv \hbar\omega_j\langle dN/dt\rangle_j(t)$, with $\langle dN/dt\rangle_j(t)$ the mean number of photons absorbed per unit of time by the atom from that wave, is given by $P_j(t) = \mathbf{E}_j(\mathbf{r}, t)\cdot d\langle\hat{\mathbf{D}}\rangle(t)/dt$. Similarly, the mean force $\mathbf{F}_j(t)$ exerted by the jth plane wave on the atom reads $\mathbf{F}_j(t) = \Sigma_{i=x,y,z}\langle\hat{D}_i\rangle(t)\nabla_{\mathbf{r}}E_{j,i}(\mathbf{r}, t)$, with $\langle\hat{D}_i\rangle$ and $E_{j,i}(\mathbf{r}, t)$ the ith components $(i = x, y, z)$ of $\langle\hat{\mathbf{D}}\rangle$ and $\mathbf{E}_j(\mathbf{r}, t)$, respectively (see, e.g., Refs. [24,25]). Of course, each plane wave does not act independently of each other on the atom, since the mean value of the atomic electric dipole moment is at any time determined by the atomic state $\hat{\rho}(t)$, which in turn is fully determined by the simultaneous action of all plane waves through the OBEs [Eq. (4)]. The total mean power absorbed from all plane waves and the net force exerted on the atom are then given by $P(t) = \Sigma_j P_j(t)$ and $\mathbf{F}(t) = \Sigma_j\mathbf{F}_j(t)$, respectively. In the RWA approximation, where one neglects the fast oscillating terms, we immediately get $P_j(t) = R_j(t)\hbar\bar{\omega}$ and $\mathbf{F}_j(t) = R_j(t)\hbar\mathbf{k}_j$, with

$$R_j(t) = \mathrm{Re}[\Omega_j(v(t) + iu(t))e^{i(\omega_j-\bar{\omega})t}]. \tag{16}$$

It also follows that $\langle dN/dt\rangle_j(t) = (\bar{\omega}/\omega_j)R_j(t)$. In the quasi-resonance condition, where $\omega_j \simeq \bar{\omega}$, $\forall\, j$, we can clearly consider $\langle dN/dt\rangle_j(t) \simeq R_j(t)$.

Within the commensurability assumption, $\omega_j - \bar{\omega} = m_j\omega_c$, $\forall\, j$. In the periodic regime, $u(t)$ and $v(t)$ are in addition T_c-periodic, and thus so are $R_j(t)$, $P_j(t)$, and $\mathbf{F}_j(t)$. In this regime, the Fourier components of $R_j(t) \equiv \sum_{n=-\infty}^{+\infty} R_{j,n}e^{\mathrm{i}n\omega_c t}$ are easily obtained by inserting the Fourier expansion [Eq. (8)] into Eq. (16). By using further Eq. (9) and $w_n = q_n w_0$ with w_0 as in Eq. (14), we get

$$R_{j,n} = \frac{\Gamma}{2}\frac{s_{j,n}}{1+s_{\mathrm{eff}}} \tag{17}$$

with $s_{j,n} = (\sigma_{j,n} + \sigma_{j,-n}^*)/2$, where

$$\sigma_{j,n} = \frac{\Omega_j}{\Gamma/2 + i(n\omega_c - \delta_j)} \sum_{l=1}^{N} \frac{\Omega_l^*}{\Gamma} q_{n+m_{lj}}. \quad \text{(18)}$$

In particular, the temporal mean value $\overline{R}_j$ of $R_j(t)$ in the periodic regime is given by the Fourier component $R_{j,0}$, and observing that $s_{j,0}$ is nothing but the parameter s_j of Eq. (15), we get $\overline{R}_j = (\Gamma/2)s_j/(1+s_{\text{eff}})$. The Fourier components of the force $\mathbf{F}_j(t) \equiv \sum_{n=-\infty}^{+\infty} \mathbf{F}_{j,n} e^{in\omega_c t}$ in the periodic regime are then given by $\mathbf{F}_{j,n} = R_{j,n}\hbar\mathbf{k}_j$ [26], and the mean force in this regime reads, consequently,

$$\overline{\mathbf{F}}_j = \frac{\Gamma}{2}\frac{s_j}{1+s_{\text{eff}}}\hbar\mathbf{k}_j, \quad \text{(19)}$$

in support of our introductory claim. Here, nevertheless, in contrast to the saturation parameter s_j of Eq. (2), the newly defined parameter s_j in Eq. (15) can be negative depending on the different phases of $\Omega_l^* q_{m_{lj}}$ with respect to Ω_j. This accounts for two important physical effects. First, it can make $\overline{R}_j$ negative, in which case the atom acts as a net mean photon emitter in the jth plane wave (stimulated emission) and the force exerted by that wave is directed opposite to $\mathbf{k}_j$ (the atom is pushed in the direction opposite to the direction of the incident photons). Second, the ratio $|s_j/(1+s_{\text{eff}})|$ can exceed 1, and the force exerted by the individual laser j can exceed the maximal spontaneous force $(\Gamma/2)\hbar k_j$, as is expected for coherent forces such as the stimulated bichromatic force [6,27].

We illustrate our formalism with the calculation of the stimulated bichromatic force in a standard four traveling-wave configuration [6]. This force is subtle, since it relies on both coherent and high-intensity effects. Except for some rough approximations, it has so far only been modeled numerically [7–12]. We consider a detuning $\delta = 10\Gamma$, a Rabi frequency amplitude of $\sqrt{3/2}\delta$, and a phase shift of $\pi/2$ for one of the waves. We show in Fig. 1 the amplitude of the resulting bichromatic force (averaged over the 2π range of the spatially varying relative phase between the opposite waves) acting on a moving atom as a function of its velocity v. As expected, the peak value of the bichromatic force is of the order of $(2/\pi)(\delta/\Gamma)$ (in units of $\hbar k\Gamma/2$) and spans a velocity range of the order of δ/Γ (in units of Γ/k) [28,29]. The narrow peaks are due to velocity-tuned "Doppleron" resonances [6]. A direct numerical integration of the OBEs completed with a numerical average in the periodic regime yields identical results.

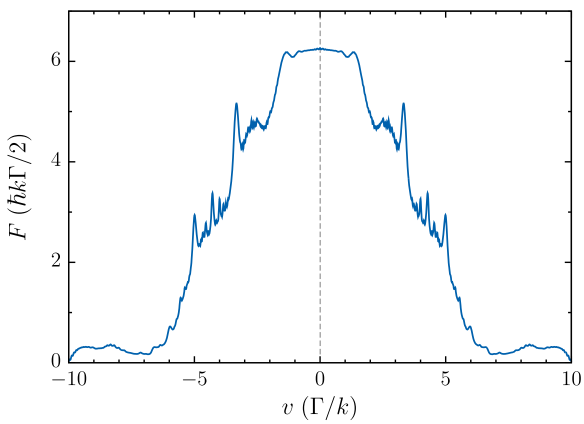

Fig. 1. Stimulated bichromatic force F as a function of the atomic velocity v computed via our formalism for a detuning $\delta = 10\Gamma$, a Rabi frequency of $\sqrt{3/2}\delta$, and a phase shift of $\pi/2$ for one laser.

3. SPECIFIC REGIMES

At low intensity, i.e., for $\Omega_{\text{T}}' \equiv \Sigma_j \Omega_{\text{R},j}/\Gamma \ll 1$, we have $|\tilde{s}| \ll 1$ and $\Sigma_{n'}|W_{n,n'}| \lesssim 2\Omega_{\text{T}}'^2 \ll 1$, $\forall n$. It implies that the resolvent $R_{W,-1} \equiv (I+W)^{-1}$ identifies to $I + \sum_{k=1}^{\infty}(-W)^k$, and the solution $\mathbf{w}$ of the infinite system [Eq. (12)] is such that $w_0 \simeq -1/[2(1+\tilde{s})] \simeq -1/2$ and $|w_n| \lesssim 2\Omega_{\text{T}}'^2$, $\forall n \neq 0$. Hence, for $m_{lj} \neq 0$, $|q_{m_{lj}}| \lesssim 4\Omega_{\text{T}}'^2 \ll 1$, and it follows that $s_j \simeq \tilde{s}_j$. If all plane waves have different frequencies, the sum over l in Eq. (11) only contains the single term $l = j$ and thus

$$s_j \simeq \frac{|\Omega_j|^2/2}{\delta_j^2 + \Gamma^2/4}. \quad \text{(20)}$$

If, in contrast, some plane waves have identical frequencies, coherent effects can be observed. However, if we are only interested in the incoherent action of the plane waves, an average $\langle\cdot\rangle_\varphi$ over all phase differences must be performed. In the low-intensity regime, the statistical delta method [30] yields $\overline{R}_j^{\text{inc}} \equiv \langle\overline{R}_j\rangle_\varphi \simeq (\Gamma/2)\langle\tilde{s}_j\rangle_\varphi/(1+\langle\tilde{s}\rangle_\varphi)$. Since the average $\langle\tilde{s}_j\rangle_\varphi$ is simply the s_j of Eq. (20), we get again $\overline{R}_j^{\text{inc}} \simeq (\Gamma/2)s_j/(1+s_{\text{eff}})$ with s_j as in Eq. (20). In all cases, the incoherent and low-intensity limit of our newly defined parameter s_j in Eq. (15) reduces to the standard expression (20), known to hold in this regime [14].

If all plane waves have the same frequency, we get $\forall j$ $s_j = \text{Re}[(\Omega_j/[\Gamma/2 - i\delta])(\Omega^*/\Gamma)]$, with $\Omega = \Sigma_l \Omega_l$ and $\delta \equiv \delta_j$. For $N = 1$, this reduces to the standard expression (20) of the saturation parameter. For two counterpropagating plane waves of identical intensity and polarization, the mean net resulting force $\overline{\mathbf{F}} = (\Gamma/2)(s_1 - s_2)/(1 + s_1 + s_2)\hbar\mathbf{k}_1$ reduces to the well-known phase-dependent dipole force in a stationary monochromatic wave $\overline{\mathbf{F}} = [4\Omega_R^2\delta\sin(\Delta\phi)]/[\Gamma^2 + 4\delta^2 + 8\Omega_R^2\cos^2(\Delta\phi/2)]\hbar\mathbf{k}_1$, with $\Delta\phi = \phi_2 - \phi_1$, and $\Omega_R \equiv \Omega_{R,1} = \Omega_{R,2}$ [5].

Very generally, in a configuration with two plane waves of different frequencies, the required solution of the infinite system [Eq. (12)] can be obtained in a continued fraction approach (see also Ref. [15] for the particular case of two waves of identical intensity and polarization in a counterpropagating configuration). For $N = 2$ and $\omega_1 \neq \omega_2$, the commensurability assumption implies that $n_2\kappa_1 = n_1\kappa_2$, with n_1 and n_2 two positive coprime integers. It follows that $\omega_c = |\omega_1 - \omega_2|/n_s$ with $n_s = n_1 + n_2$, $m_1 = \text{sgn}(\omega_1 - \omega_2)n_2$, $m_2 = \text{sgn}(\omega_2 - \omega_1)n_1$, $m_{12} = \text{sgn}(\omega_1 - \omega_2)n_s$, and $M_0 = \{\pm n_s\}$. The infinite system [Eq. (10)] reads in this case, $\forall n$,

$$w_n + \mathcal{W}_{n,n_s}w_{n+n_s} + \mathcal{W}_{n,-n_s}w_{n-n_s} = \frac{-1}{2(1+\tilde{s})}\delta_{n,0}. \quad \text{(21)}$$

The system only couples together the Fourier components w_n with $n = kn_s$ $(k \in \mathbb{Z})$. All other components are totally decoupled from these former components and thus vanish, since they satisfy a homogeneous system. Hence, the only *a priori* nonvanishing ratios q_n and Fourier components $R_{j,n}$ and $\mathbf{F}_{j,n}$ are for these specific values of n, and the periodic regime is rather characterized by the beat period $T_c/n_s = 2\pi/|\omega_1 - \omega_2|$. For $n \neq 0$, Eq. (21) implies $w_n/w_{n-n_s} = -\mathcal{W}^*_{-n,n_s}/[1 + \mathcal{W}_{n,n_s}(w_{n+n_s}/w_n)]$. Applying recursively this relation for $n = n_s, 2n_s, 3n_s, \ldots$ yields $q_{n_s} = -\mathcal{W}^*_{-n_s,n_s}/[1 + \mathcal{K}^\infty_{k=1}(p_k/1)]$, with $p_k = -\mathcal{W}_{kn_s,n_s}\mathcal{W}^*_{-(k+1)n_s,n_s}$ and $\mathcal{K}^\infty_{k=1}(p_k/1)$ the continued fraction $p_1/(1 + p_2/(1 + p_3/\ldots))$. Dividing further Eq. (21) for $n = kn_s$ $(k > 0)$ by w_0 yields the recurrence relation $q_{(k+1)n_s} = -[q_{kn_s} + \mathcal{W}^*_{-kn_s,n_s}q_{(k-1)n_s}]/\mathcal{W}_{kn_s,n_s}$, which allows for computing all remaining q_{kn_s} with $k > 1$, and hence along with q_{n_s} all nonzero Fourier components R_{j,kn_s} and $\mathbf{F}_{j,kn_s}$ $(k \geq 0)$. The set $\{\mathcal{W}_{kn_s,n_s}, k \in \mathbb{Z}\}$ proves to be independent of the actual value of n_s, and thus, so is the set of numbers $\{q_{kn_s}, k \in \mathbb{Z}\}$. The continued fraction and, as expected, all Fourier components with respect to the beat periodicity are independent of the arbitrary choice of the weighting factors κ_1 and κ_2 that set n_s.

4. CONCLUSION

In conclusion, we have provided a unified and exact formalism to calculate within the usual RWA the mechanical action of a set of arbitrary plane waves acting simultaneously on a single two-level atom. We have shown how to write the steady mean light forces acting in strict generality in the form of Eq. (2), including coherent and high-intensity effects, with a generalized definition of the parameter s_j [Eq. (15)]. We have provided within a commensurable assumption similar expressions for all Fourier components of the light forces in the steady periodic regime that is established after a transient. These results provide a novel tool for engineering the mechanical action of lasers on individual atoms. They can offer an alternative to purely numerical approaches, where the extraction of the mean force and their Fourier components can be subjected to instabilities, especially when the forces vary very slowly. Our results always converge to the exact values. In the case of two lasers, we have shown in strict generality how to convert the limit of Eq. (13) into a continued fraction computable straightforwardly. This work admits a natural extension where multilevel atoms are instead considered to account for the Zeeman degeneracy of the atomic levels.

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